FILLING FAMILIES AND STRONG PURE INFINITENESS

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Abstract. We introduce filling families with the matrix diagonalization property of Definition 4.6 to extend and refine the work by Rørdam and the first named author in [26]. They allow to establish transparent necessary and sufficient criteria for strong pure infiniteness of crossed products of C^*-algebras by group actions as obtained in [28]. Here we use particular filling families to improve a result on "local" pure infiniteness in [7] and show that the minimal tensor product of a strongly purely infinite C^*-algebra and a exact C^*-algebra is again strongly purely infinite. Finally we derive with help of suitable filling families an easy sufficient criterion for the strong pure infiniteness of crossed products $A \rtimes_\varphi \mathbb{N}$ by an endomorphism φ of A (cf. Theorem 7.9). Our work confirms that the special class of nuclear Cuntz-Pimsner algebras constructed in [19] consist of strongly purely infinite C^*-algebras, and thus absorb $\mathcal{O}_\infty$ tensorially.

Contents

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1. Introduction

The classification program of G. Elliott for nuclear C^*-algebras [17, 37], has been an active field of research for more than 40 years, beginning with the classification of AF-algebras by Bratteli [8] and Elliott [15]. This paper focuses how one might verify when C^*-algebras are strongly purely infinite, a property which is necessary for classification of separable nuclear C^*-algebras with the help of an ideal system equivariant version of KK-theory.

In Section 2, following a short Section 3 on our notation and preliminary results, we familiarise the reader with the notion of strongly purely infinite C^*-algebras A. One formulation of this property (see Remark 3.2) is that for each given pair of positive elements $a_1, a_2 \in A$, any $c \in A$ and $\varepsilon \geq \tau > 0$ there exist elements $s_1, s_2 \in A$ such that

$$\|s_1^* a_1 s_1 - a_1\| < \varepsilon\,, \quad \|s_2^* a_2 s_2 - a_2\| < \varepsilon \quad \text{and} \quad \|s_1^* c s_2\| < \tau\,. \tag{1}$$

We discus a number of different formulations, relate the notion of strong pure infiniteness to other similar notions, and perhaps most importantly connect it to $\mathcal{O}_\infty$ absorption, classification of non-simple C^*-algebras and previous work in [12, 22, 23, 24, 26, 42, 43] among others.

In Section 4 we introduce the notion of a filling family and a family with the matrix diagonalization property. The first notion is roughly speaking a intrinsic property encoding a certain ideal structure for a C^*-algebra (for a C^*-subalgebra $B \subseteq A$ the map $I \mapsto I \cap B$ from ideals in the C^*-algebra A to ideals in B is injective if the positive element in B is a filling family for A, see Remark 4.4(ii)). The later notion is a weakening of strong pure infiniteness where we look at solutions of the inequality (1) but only for a specified family of positive elements a_1, a_2 in A. We prove the following result.

Theorem 1.1. *Suppose that A_+ contains a filling family $\mathcal{F}$ (Def. 4.2), that has the diagonalization property in A (Def. 4.7). Then A is strongly purely infinite.*

In Section 5 we develop tools for the verification the matrix diagonalization property. The properties that we study are of the following type: Given subsets $\mathcal{F} \subseteq A_+$, $\mathcal{C} \subseteq A$ and $\mathcal{S} \subseteq A$. Suppose for each given $a_1, a_2 \in \mathcal{F}$, $c \in \mathcal{C}$ and $\varepsilon \geq \tau > 0$ there exist elements $s_1, s_2 \in \mathcal{S}$ that fulfill (1).

Question 1.2. Under which conditions on $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{S}$ can the inequalities (1) be solved by some $s_1, s_2 \in \mathcal{S}$ for given $(a_1, a_2, c, \varepsilon \geq \tau > 0)$ with $a_1, a_2 \in \mathcal{F}$, but with more general elements c?

We show (as a special case of Lemma 5.6) that the inequalities (1) can be solved for any c in the closure of the linear span of $\mathcal{C}$ provided that (i) $\mathcal{F}$ is invariant under ε-cut-downs, (ii) $\mathcal{S}$ is a (multiplicative) sub-semigroup of A and (iii) for every $s, s_1, s_2 \in \mathcal{S}$, $\varphi \in \mathrm{C}_c(0,\infty]_+$ and $c \in \mathcal{C}$

$$\varphi(a_1)s \in \mathcal{S}, \quad \varphi(a_2)s \in \mathcal{S}, \quad s_2^*\mathcal{C}s_1 \subseteq \mathcal{C}, \quad \text{and } \varphi(a_1)c\varphi(a_2) \in \mathcal{C}.$$

Our results are more general as applications require the study of families $\mathcal{F}$ that are not necessarily invariant under ε-cut-downs.

In Section 6 we consider our first application: tensor products. By invoking on the work in [7] we show the following result (where it does not matter which of the two algebras is exact):

Theorem 1.3. *The minimal tensor product $A \otimes^{min} B$ of a C^*-algebra A and an exact C^*-algebra B is strongly purely infinite if at least one of A or B is a strongly purely infinite C^*-algebra.*

In Section 7 we consider our second application: endomorphism crossed products. We begin the section by introducing the action $\sigma\colon \mathbb{Z} \to \mathrm{Aut}(A_e)$ associated to φ, which is the corresponding action of the integers $\mathbb{Z}$ on the inductive limit A_e of the sequence $A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$. We then require mild sufficient properties (ND) and (CP) ensuing that $A \rtimes_\varphi \mathbb{N}$ can be naturally identified with a hereditary C^*-subalgebra of $A_e \rtimes_\sigma \mathbb{Z}$. The properly (ND) is automatic when φ is injective and the second property (CD) ensures that the canonical map $A \to A_e$ extends to a strictly continuous *-homomorphism $\mathcal{M}(A) \to \mathcal{M}(A_e)$ of the multiplier algebras, cf. Lemma 7.2. We prove the following result (as a special case of Theorem 7.9 where it is only required that b_1, b_2, c belongs to a dense φ-invariant C^*-local *-subalgebra of A):

Theorem 1.4. *Let φ be an endomorphism of a separable C^*-algebra A satisfying properties (ND) and (CP). Suppose that σ is residually properly outer (Def. 7.8) and that for every $b_1, b_2, c \in A$ and $\varepsilon > 0$ there exist k, n_1, $n_2 \in \mathbb{N} \cup \{0\}$ and s_1, $s_2 \in A$ such that $\| s_j^* \varphi^k(b_j^* b_j)\, s_j - \varphi^{n_j}(b_j^* b_j) \| < \varepsilon$ for $j = 1, 2$ and $\| s_1^* \varphi^k(c) s_2 \| < \varepsilon$. Then $A_e \rtimes_\sigma \mathbb{Z}$ and $A \rtimes_\varphi \mathbb{N}$ is strongly purely infinite.*

We end by looking at certain class of Cuntz-Pimsner algebras. More specifically we look at *-monomorphisms $h\colon C \hookrightarrow \mathcal{M}(C)$ of stable nuclear separable σ-unital C^*-algebras C to which we associate its canonical Hilbert bi-module $\mathcal{H}(h,C)$. It turns out that for many cases of interest – identifiable in terms of conditions on h – the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{H}(h,C))$ is strongly purely infinite and hence tensorially absorb $\mathcal{O}_\infty$ (see Remark 8.3). We prove these result, previously shown in [19], by identifying these Cuntz-Pimsner algebras as endomorphism crossed products.

2. Notation and preliminary results

Let A_+ denote the positive elements of A, and $a_+ := (|a|+a)/2 \in A_+$ and $a_- := (|a|-a)/2) \in A_+$ the positive and negative parts of an selfadjoint element a of A, where $|a| := (a^*a)^{1/2}$. If $a \in A_+$, then $(a-\varepsilon)_+$, i.e., the positive part of $a - \varepsilon 1 \in \mathcal{M}(A)$, is again in A_+ itself. Here $\mathcal{M}(A)$ is the *multiplier algebra* of A. This notation will be used also for functions $f\colon \mathbb{R} \to \mathbb{R}$, then e.g. $(f-\varepsilon)_+(\xi) = \max(f(\xi) - \varepsilon\,, 0)\,$. Clearly, $\chi((a-\varepsilon)_+) = (\chi(a)-\varepsilon)_+$ for each character χ on the C^*-subalgebra $C^*(a) \subseteq A$ generated by a. This implies for all $a \in A_+$ and $b \in A$ that

$$\|(a-\varepsilon)_+\| = (\|a\| - \varepsilon)_+ \quad \text{and} \quad \|b - (a-\varepsilon)_+\| \le \|b-a\| + \varepsilon\,.$$

A subset $\mathcal{F} \subseteq A_+$ is invariant under *ε-cut-downs* if for each $a \in \mathcal{F}$ and $\varepsilon \in (0, \|a\|)$ we have $(a-\varepsilon)_+ \in \mathcal{F}$. The minimal unitalization of A is denoted $\tilde{A}$. Restriction of a map f to X is denoted $f|X$. We let $\mathrm{C}_c(0,\infty]_+$ denote the set of all non-negative continuous functions φ on $[0,\infty)$ with $\varphi|[0,\eta] = 0$ for some $\eta \in (0,\infty)$, such that $\lim_{t\to\infty}\varphi(t)$ exists.

Remarks 2.1. (i) Suppose that $a, b \in A_+$ and $\varepsilon > 0$ satisfy $\|\,a-b\,\| < \varepsilon\,$. Then $(b-\varepsilon)_+ \in A$ can be decomposed into $d^*ad = (b-\varepsilon)_+$ with some *contraction* $d \in A$ ([26, lem. 2.2]).

(ii) Let $\tau \in [0,\infty)$ and $0 \le b \le a + \tau \cdot 1$ (in $\mathcal{M}(A)$), then for every $\varepsilon > \tau$ there is a *contraction* $f \in A$ such that $(b-\varepsilon)_+ = f^*a_+f$. (See [26, lem. 2.2] and [7, sec. 2.7].)

(iii) Let $a, b, y \in A$, $\delta > 0$. Let φ denote the continuous function with compact support in $(0,1]$ given by $\varphi(t) := \min(1, (2/\delta)(t-\delta/2)_+)$ and the $q := q(y,\delta) \in A$ be the contraction $q(y,\delta) := \varphi(yy^*)v = v\varphi(y^*y)$, where $v|y| = y$ is the polar-decomposition of y in A^{**}. Then:

(1) $q^*(yy^* - \mu)_+q = (y^*y - \mu)_+$ for all $\mu \ge \delta\,$.

(2) $0 \le a \le b$ and $ab = a$ imply $a(b - 1/2)_+ = a/2$.

(iv) A matrix $[b_{k\ell}] \in M_2(A)$ is positive, if and only if, $b_{11}, b_{22} \in A_+$, $b_{21} = b_{12}^*$ and the transformation $(b_{11} + 1/k)^{-1/2}\, b_{12}\, (b_{22} + 1/k)^{-1/2}$ is a contraction for every $k \in \mathbb{N}$. If $[b_{k\ell}] \in M_2(A)_+$, then $b_{12} = \lim_{k\to\infty} b_{11}^{1/2}\, (b_{11} + 1/k)^{-1/2}\, b_{12}\, (b_{22} + 1/k)^{-1/2}\, b_{22}^{1/2}$.

If $[a_{ij}] \in M_n(A)_+$, $n \ge 2$, then the 2×2-matrices $[b_{k\ell}] \in M_2(A)$ with $b_{11} := a_{ii}$, $b_{22} := a_{jj}$ and $b_{21}^* = b_{12} := a_{ij}$ are positive for each $i \ne j$. In particular, $a_{ij} \in \overline{a_{ii} A a_{jj}}$, and $a_{ij} = \lim_{k\to\infty} b_i^{(k)} a_{ij} b_j^{(k)}$ for the contractions $b_j^{(k)} := (a_{jj} + 1/k)^{-1/2} a_{jj}^{1/2}$.

We omit the proofs of (i)–(iv): They are cited or can be checked easily.

3. Strongly purely infinite C^*-algebras

Recent classification theory for C^*-algebras in the bootstrap class cf. [5, chp. 9.23] extends partly to non-simple algebras (cf. [13, 14, 22, 30, 35, 38]). The classification of non-simple nuclear C^*-algebras requires to take in account the structure of the primitive ideal spaces. If we classify algebras with the help of an ideal system equivariant version of KK-theory, then we can not distinguish an algebra A from $A \otimes \mathcal{O}_\infty$. This is because one can tensor the ideal system equivariant KK-equivalences with ordinary KK-equivalences of nuclear algebras ([1]) and then one can use that $\mathcal{O}_\infty$ is KK-equivalent to the complex numbers $\mathbb{C}$ ($\mathcal{O}_\infty$ and $\mathbb{C}$ have the same K-theory by [11, cor. 3.11] and are in the bootstrap class, hence are KK-equivalent). Thus, the class of algebras suitable for such a classification contains only nuclear separable C^*-algebras that absorb $\mathcal{O}_\infty$ tensorially.

The requirement $A \otimes \mathcal{O}_\infty \cong A$ looks like a simple criterium, but is difficult to verify, e.g. for crossed products. An intrinsic characterization of $\mathcal{O}_\infty$ absorbing nuclear separable C^*-algebras motivated the following notion of *strongly purely infinite* algebras:

Definition 3.1. A C^*-algebra A is *strongly purely infinite* (for short: *s.p.i.*) if, for every $a, b \in A_+$ and $\varepsilon > 0$, there exist elements $s, t \in A$ such that

$$\| s^* a^2 s - a^2 \| < \varepsilon\,, \ \| t^* b^2 t - b^2 \| < \varepsilon \ \text{ and } \ \| s^* a b t \| < \varepsilon\,. \tag{2}$$

It was shown in [26] that every $\mathcal{O}_\infty$ absorbing C^*-algebra is strongly purely infinite. If the C^*-algebra A is separable, nuclear and strongly purely infinite then, conversely, A tensorially absorbs $\mathcal{O}_\infty$ (cf. [26] for the cases of stable or unital algebras, and [23,

[1] See [40, prop. 2.4(b)] for the non-equivariant case, the equivariant case is similar.

cor. 8.1] for the general case, see also [42, 43] or [24, prop. 4.4(5), rem. 4.6] for other proofs of the general case). There exist strongly purely infinite non-nuclear stable simple separable C*-algebras A, that are not isomorphic to $A \otimes \mathcal{O}_\infty$, cf. [12]. See also [27, exp. 4.6] for an example of a simple and exact crossed product $A \rtimes G$ of a type I C^*-algebra A by the exact group $G := F_2 \times \mathbb{Z}$ safisfying that $A \rtimes G$ is strongly purely infinite but does not absorb $\mathcal{O}_\infty$.

The nuclearity of the algebra is not a natural assumption for the study of strong pure infiniteness, because proofs for KK-classification use corona algebras or asymptotic algebras, that are even not exact for not sub-homogenous algebras ([2]), but are still strongly purely infinite in the sense of Definition 3.1. Fortunately, multiplier algebras, stable coronas and asymptotic algebras of strongly purely infinite σ-unital algebras are again strongly purely infinite.

The very basic result for the classification program is the embedding result for exact algebras into strongly purely infinite algebras, cf. [22]. In this way the notion of strongly purely infinite algebras is of importance for the classification program. This explains our desire to find methods and criteria that allow to check if a given class of (not necessarily simple) algebras are purely infinite in the *strong* sense of Definition 3.1.

It has been realized in an early stage of the classification of – simple – pi-sun algebras that many of those algebras are stably isomorphic to crossed product of boundary actions of hyperbolic groups [2, 29, 20] or as corner-endomorphism cross-product C^*-algebras [9, 10] and its generalizations. Therefore it is likely that criteria for strong pure infiniteness of crossed products can be helpful to detect also the range of KK-classification of non-simple C^*-algebras.

Remarks 3.2. (i) It was shown in the proof of [26, cor. 7.22] – but not mentioned in its formulation – that the Definition 3.1 of strong pure infiniteness implies that for each $a, b \in A_+$ and $c \in A$ there exist contractions $s_1, s_2 \in A_\omega$ with $s_1 a = a s_1$, $s_2 b = b s_2$, $s_1^* s_1 a = a$, $s_2^* s_2 b = b$ and $s_1^* c s_2 = 0$. In particular we have that our Definition 3.1 of strongly purely infinite C^*-algebras is equivalent to the formally stronger requirement, that for each $a, b \in A_+$, $c \in A$ and $\varepsilon > 0$, there exist *contractions* $s, t \in A$ such that

$$\| s^* a s - a \| < \varepsilon\,, \ \| t^* b t - b \| < \varepsilon \ \text{ and } \ \| s^* c t \| < \varepsilon\,. \tag{3}$$

[2] It is because $\mathcal{L}(\ell_2)$ is a C^*-subquotient of each not sub-homogenous sub-Stonean algebra.

The proof of [26, cor. 7.22] contains some typos ([3]). Compare also the proof of the implication (s.p.i.)$\Rightarrow$(I) in [23, thm. 4.1].

(ii) The proofs of [26, cor. 7.22] and of [23, thm. 4.1] giving contractions s, t in inequalities (3) both use a fairly deep local version of a "generalized Weyl-von Neumann theorem" [26, thm. 7.21]. But if we apply our Lemma 5.4(iii) to A and $\mathcal{F} := A_+$ then we get at least the following bounds: *A C^*-algebra A is strongly purely infinite in the sense of Definition 3.1, if and only if, for every $a_1, a_2 \in A_+$, $c \in A$ and $\varepsilon \geq \tau > 0$, there exists $s_1, s_2 \in A$ that satisfy the inequalities (1) and have norms that satisfy $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$.*

(iii) If we take $a = b$ in the Definition 3.1, then the inequalities show that each $a^2 \in A_+$ is properly infinite in A, see [26, prop. 5.4]. Hence A is purely infinite in sense of [25, def. 4.1] by [25, thm. 4.16]. In general it is an open question whether the notions of strong pure infiniteness, pure infiniteness and weak pure infiniteness coincide or not. When A is simple the three properties are equivalent. We refer to [7] and [26] for other special cases where weak and strong pure infiniteness coincide.

(iv) It should be noticed that the original definition of J. Cuntz of purely infinite C^*-algebras in [11] coincides only in some special cases – e.g. for simple algebras – with the definition of purely infinite C^*-algebras in [25].

Also [29, thm. 9] does not show pure infiniteness for crossed products coming from *local* boundary actions [29] – even not in the sense of [25, def. 4.1]. Both these definitions in [11, 25] are still not suitable for the classification in general – except in combination with other assumptions, like e.g. tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$.

4. Filling families and strong pure infiniteness

A suitable algebraic theory for invariants of strongly purely infinite C^*-algebras is not in sight, different to the property of pure infiniteness of C^*-algebras A that is equivalent to $2[a] = [a]$ in the Cuntz semigroup for every $a \in (A \otimes \mathcal{K})_+$. The absence of a reasonable algebraic description forces us to develop new methods to detect and describe strong pure infiniteness.

[3] Replace in [26] "$d \in B_\omega$" by "$d \in M_2(A)_\omega$" and "D of B_ω" by "D of $M_2(A)_\omega$" in line +14 on page 252, "$B_\omega = M_2((B_0)_\omega)$" by "$M_2(A)_\omega = M_2(A_\omega)$" in line -6 on page 252, and "contractions in $B_0 \subseteq A$" by "contractions in A" in line +11 on page 253.

Here we introduce two new concepts: Firstly we work with the idea of a *filling* family $\mathcal{F} \subseteq A_+$, cf. Definition 4.2. Secondly we introduce the notion of a family $\mathcal{F} \subseteq A_+$ with the *matrix diagonalization property* as a refinement of the notion of the matrix diagonalzation property introduced in [26, def. 5.5] (cf. Definition 4.6). We say "family" because we use the elements of $\mathcal{F}$ mainly to form a family of selfadjoint $n \times n$-matrices with diagonal entries from $\mathcal{F}$ for $n = 2, 3, \ldots$ – together with certain restrictions on the off-diagonal entries.

Before defining a filling family we need a lemma. Notice that one can replace in part (ii) of the following Lemma 4.1 *primitive* ideals by *all closed* ideals I with $D \not\subseteq I$, because every closed ideal is the intersection of primitive ideals.

Lemma 4.1. *Let $\mathcal{F}$ be a subset of A_+. The following properties of $\mathcal{F}$ are equivalent:*

(i) *For every $a, b, c \in A$ with $0 \le a \le b \le c \le 1$, with $ab = a \neq 0$ and $bc = b$, there exists $z_1, z_2, \ldots, z_n \in A$ and $d \in A$ with $z_j(z_j)^* \in \mathcal{F}$, such that $ec = e$ and $d^*ed = a$ for $e := z_1^*z_1 + \ldots + z_n^*z_n$.*

(ii) *For every hereditary C^*–subalgebra D of A and every primitive ideal I of A with $D \not\subseteq I$ there exist $f \in \mathcal{F}$ and $z \in A$ with $z^*z \in D$ and $zz^* = f \notin I$.*

Proof. (i)⇒(ii): Since $D \not\subseteq I$, there exists $g \in D_+$ with $\|g\| = \|g + I\| = 3$. Let $a := (g-2)_+$, $b := (g-1)_+ - (g-2)_+$ and $c := g - (g-1)_+$. Then $a \notin I$, $0 \le a \le b \le c \le 1$, $ab = a$, $bc = c$, $g = c + b + a$ and $\|c + b\| \le 2$. By (i), we find $z_1, z_2, \ldots, z_n \in A$ and $d \in A$ with $z_j(z_j)^* \in \mathcal{F}$, such that $e := z_1^*z_1 + \ldots + z_n^*z_n$ satisfies $ec = e$ and $d^*ed = a$. It follows that $e \in cAc \subseteq D$ and $e \notin I$. Hence $(z_j)^*z_j \in D \backslash I$ for some $j \in \{1, \ldots, n\}$. Then $z := z_j$ and $f := z_j(z_j)^* \in \mathcal{F}$ satisfy (ii).

(ii)⇒(i): Suppose that $a, b, c \in A$ with $0 \le a \le b \le c \le 1$, $ab = a \neq 0$, $bc = b$ are given. Let $D := \overline{bAb}$ and let $\mathcal{Z}$ denote the set of all $z \in A$ with $zz^* \in \mathcal{F}$ and $z^*z \in D$. It follows that $z^*zc = z^*z$ and $zu \in \mathcal{Z}$ for $z \in \mathcal{Z}$ and every unitary u in the minimal unitization $\widetilde{D}$ of D. Consider the set M of $d \in D_+$ with the property that there are $z_1, \ldots, z_n \in \mathcal{Z}$ and a $\rho \in (0, \infty)$ with

$$0 \le d \le \rho \cdot (z_1^*z_1 + \ldots + z_n^*z_n)\,.$$

Clearly, M is a (not necessarily closed) hereditary convex cone in D_+, and $u^*Mu \subseteq M$ for every unitary u in $\widetilde{D}$. Arguments in the proof of [33, thm. 1.5.2] show that the set $L(M) := \{d \in D\,;\; d^*d \in M\}$ is a (not necessarily closed) left ideal of D. Since each element of an unital C^*-algebra is the linear combination of unitaries in this algebra,

we get that $dx \in L(M)$ for each $x \in \widetilde{D}$, $d \in L(M)$. In particular, $L(M)$ is a two-sided ideal of D. It follows, that the closure $J := \overline{L(M)}$ of the two-sided ideal $L(M)$ is an ideal of D. The closure $K := \overline{M}$ of M is again a hereditary convex cone in D_+, with $L(K) = J$. Thus, K is the positive part J_+ of the closed left ideal J of D.

Since D is hereditary, the closed linear span I of AJA is a closed ideal I of A with $J = D \cap I$, see [6, II.5.3.5].

The property (ii) implies $D \subseteq I$, because $D \cap I$ contains *all* elements z^*z with $z \in \mathcal{Z}$ (since $z^*z \in M$ for $z \in \mathcal{Z}$). We obtain that $D_+ = J_+ = \overline{M}$. In particular, $b \in \overline{M}$, and we find an element $g \in M$ with $b \leq g + 1/4$. By definition of M there are $z_1, \ldots, z_n \in \mathcal{Z}$ and $\rho \in (0, \infty)$ with $g \leq \rho e$ for $e := z_1^* z_1 + \ldots + z_n^* z_n$. Then $e \in D_+$ and $ec = e$ by definitions of D and of $\mathcal{Z}$. There is a contraction $d_1 \in D$ with $(b - 1/2)_+ = \rho d_1^* e d_1$ by Remark 2.1(ii). It follows $a = d^* e d$ for $d := \sqrt{2\rho} \cdot d_1 a^{1/2}$, because $a(b - 1/2)_+ = a/2$. by Remark 2.1(iii,2). □

Definition 4.2. Let $\mathcal{F}$ be a subset of A_+. The set $\mathcal{F}$ is a *filling family* for A, if $\mathcal{F}$ satisfies the equivalent conditions (i) and (ii) of Lemma 4.1.

Remarks 4.3. (i) Lemma 4.1(ii) shows that $\mathcal{F} := A_+$ is a filling family for A.

(ii) We warn the reader that our fundamental notion of "filling families" does not require an existence of some type of actions, e.g. group actions. In particular, they have nothing in common with the notion of n-filling actions in [20].

(iii) If $\mathcal{F} \subseteq A_+$ is *invariant under ε-cut-downs*, i.e., if $(a - \varepsilon)_+ \in \mathcal{F}$ for each $a \in \mathcal{F}$ and $\varepsilon \in (0, \|a\|)$, then we can replace the Murray–von-Neumann equivalence $z^*z \approx_{MvN} f$ in Lemma 4.1(ii) by Cuntz equivalence (denoted by $\sim$) in Definition 4.2(i):
For every hereditary C^*–subalgebra D of A and every primitive ideal I of A with $D \not\subseteq I$ there exist $f \in \mathcal{F} \setminus I$ and $g \in D$ with $g \sim f$.

Remarks 4.4. It is in general not easy to check if a family $\mathcal{F} \subseteq B_+$ is filling for B or not. We list some cases $\mathcal{F} \subseteq A_+ \subseteq B$, where $A \subseteq B$ are C^*-algebras:

(i) If $A = \mathrm{C}_0(X)$, then $\mathcal{F} \subseteq A_+$ is filling, if and only if, the supports of the functions $f \in \mathcal{F}$ build a base of the topology of X.
(ii) If $\mathcal{F} := A_+ \subseteq B$ is filling for B, then the map $I \in \mathcal{I}(B) \mapsto I \cap A \in \mathcal{I}(A)$ is injective, i.e., A separates the closed ideals of B, cf. [39].

(iii) Let R denote a nuclear, separable, simple and finite C^*-algebra such that $B := M_2(R)$ is properly infinite (cf. Rørdam [36]). Thus, there exist a unital *-homomorphism $\varphi\colon \mathcal{O}_\infty \to B$. Consider the image $A := \varphi(\mathcal{O}_\infty)$ and let $\mathcal{F} := A_+$. Then $\mathcal{F}$ separates the ideals $\{0\}$ and B of B, but $\mathcal{F}$ is not filling for B.

(iv) If $D \neq \mathbb{C}$ is a simple, unital, and *stably finite* C^*-algebra, X is a locally compact Hausdorff space, $B := \mathrm{C}_0(X) \otimes D$ and $A := \mathrm{C}_0(X) \otimes 1 \subseteq B$, then $\mathcal{F} := A_+$ separates the ideals of B, but is not filling for B.

(v) Let $A := \mathbb{C} \oplus \mathbb{C}$, $\sigma(u, v) := (v, u)$, $\mathcal{F} := \{(1, 0)\} \subseteq A_+$, and $B := A \rtimes_\sigma \mathbb{Z}_2 \cong M_2(\mathbb{C})$. Then $\mathcal{F}$ and A_+ are filling for B, but $\mathcal{F}$ is not filling for A.

(vi) The example in [4, p.123] has the property that $\mathcal{K} \otimes A_{1-\theta} \cong \mathcal{K} \rtimes \mathbb{Z}^2$ is simple and stably finite and $\mathcal{F} := \mathcal{K}_+ \subseteq \mathcal{K} \rtimes \mathbb{Z}^2$ is not filling for $\mathcal{K} \rtimes \mathbb{Z}^2$.

All above examples are refereed or easily verified. For the convenience of the reader we give some hint for part (vi): The only primitive ideal of $\mathcal{K}\rtimes\mathbb{Z}^2$ is $\{0\}$. Take a minimal projection $p \in \mathcal{K}_+ \subset \mathcal{K}\rtimes\mathbb{Z}^2$. Then we find a non-zero projection $q \in p(\mathcal{K}\rtimes\mathbb{Z}^2)p$ that is not MvN-equivalent to p. Let $D := q(\mathcal{K}\rtimes\mathbb{Z}^2)q$. Then there is no non-zero contraction $z \in \mathcal{K}\rtimes\mathbb{Z}^2$ with $z^*z \in \mathcal{K}$ and $zz^* \in D$, because p is not infinite.

Lemma 4.5. *Suppose that $A \subseteq B$ is a C^*-subalgebra of B and $\mathcal{F} \subseteq A_+$ is a subset of A_+. If $\mathcal{F}$ is filling for A, and A_+ is filling for B, then $\mathcal{F}$ is a filling family for B.*

Proof. Let $D \subseteq B$ hereditary, $I \subseteq B$ closed ideal with $D \not\subseteq I$. By assumption, there is $z \in B$ with $z^*z \in D$, $zz^* \notin I$ and $zz^* \in A_+$. Let E denote the hereditary C^*-subalgebra of A generated by zz^*, i.e., $E := \overline{zz^*Azz^*}$. Since $zz^* \notin I$, the algebra E is not contained in the ideal $J := A \cap I$ of A. By assumption, there exists $y \in A$ with $yy^* \in \mathcal{F}$, $y^*y \in E \subseteq A$ and $yy^* \notin J$. Let $v(z^*z)^{1/2} = z$ denote the polar decomposition of z in B^{**}. Then $x := yv \in B^{**}$ satisfies $x \in B$, because $y^*y \in \overline{zz^*Bzz^*}$. Moreover, $xx^* \notin I$, $xx^* \in \mathcal{F}$ and $x^*x \in v^*Ev \subseteq D$: To see $xx^* \in \mathcal{F}$, notice that $zz^*vv^* = zz^*$, hence for all $e \in E$, $evv^* = e$. Since $y^*y \in E$, we get $\|yvv^* - y\|^2 = 0$, so $xx^* = yvv^*y^* = yy^* \in \mathcal{F}$. □

Definition 4.6. Let $\mathcal{S} \subseteq A$ be a multiplicative sub-semigroup of a C^*-algebra A and $\mathcal{C} \subseteq A$ a subset of A. An n-tuple $(a_1, \ldots, a_n)$ of positive elements in A has the *matrix diagonalization property with respect to $\mathcal{S}$ and $\mathcal{C}$*, if for every $[a_{ij}] \in M_n(A)_+$ with $a_{jj} = a_j$ and $a_{ij} \in \mathcal{C}$ (for $i \neq j$) and $\varepsilon_j > 0, \tau > 0$ there are elements $s_1, \ldots, s_n \in \mathcal{S}$ with

$$\|s_j^* a_{jj}\, s_j - a_{jj}\| < \varepsilon_j\,, \quad \text{and} \quad \|s_i^* a_{ij} s_j\| < \tau \ \text{ for } i \neq j\,. \tag{4}$$

If $\mathcal{S} = \mathcal{C} = A$ then this is the *matrix diagonalization property* of $(a_1, \ldots, a_n)$ as defined in [26, def. 5.5], and we say that $(a_1, \ldots, a_n)$ has matrix diagonalization (in A).

Definition 4.7. Let $\mathcal{F}$ be a subset of A_+. The family $\mathcal{F}$ has the *(matrix) diagonalization property* (in A) if each finite sequence $a_1, \ldots, a_n \in \mathcal{F}$ has the matrix diagonalization property (in A) of Definition 4.6.

Remarks 4.8. (i) By Remark 2.1(i) and a preceding inequality, it follows that the n-tuple $(a_1, \ldots, a_n)$ has the matrix diagonalization with respect to A and $\mathcal{C}$ if, and only if, for each $[a_{ij}] \in M_n(A)_+$ with $a_{jj} = a_j$ and $a_{ij} \in \mathcal{C}$ (for $i \neq j$) and $\varepsilon_j > 0, \tau > 0$ there are elements $s_1, \ldots, s_n \in A$ that satisfy the equations and inequalities

$$s_j^* a_{jj}\, s_j = (a_{jj} - \varepsilon_j)_+\,, \quad \text{and} \quad \|s_i^* a_{ij} s_j\| < \tau \ \text{ for } i \neq j\,. \tag{5}$$

(ii) If we replace the ε_j and τ in inequalities (4) by $\varepsilon := \min(\,\varepsilon_1\,,\,\ldots\,,\,\varepsilon_n\,,\,\tau\,)$, then this new definition is the same as Definition 4.6 with $\varepsilon_1 = \cdots = \varepsilon_n = \tau = \varepsilon$. But the latter is an equivalent formulation of Definition 4.6.

(iii) The following is again equivalent to the matrix diagonalization property: The n-tuple $(a_1, \ldots, a_n)$ has the matrix diagonalization property with respect to $\mathcal{S}$ and $\mathcal{C}$, if and only if, for each positive matrix $[a_{ij}] \in M_n(A)$ with diagonal entries $a_{jj} = a_j$ and $a_{ij} \in \mathcal{C}$ (for $i \neq j$), there exists a sequences $s^{(k)} \in M_n(A)$, $k = 1, 2, \ldots$, of diagonal matrices $s^{(k)} = \operatorname{diag}(s_1^{(k)}, \cdots, s_n^{(k)})$ with $s_j^{(k)} \in \mathcal{S}$, such that

$$\lim_{k\to\infty} \|\,(s^{(k)})^*\,[\,a_{ij}\,]\,s^{(k)} - \operatorname{diag}(a_1, \ldots, a_n)\,\| \;= 0\,.$$

(iv) It is important for our applications to find an estimate of $\max_j \|s_j\|^2$ depending only on $\min(\,\varepsilon_1\,,\,\ldots\,,\,\varepsilon_n\,)$ that does not depend on $\{a_{ij}\,;\ j \neq i\}$ or on $\tau > 0$. Therefore, we often use (starting from proof of Lemma 4.9) the equivalent formulation of Definition 4.6 with values $\varepsilon_j := \varepsilon > 0$ and independent $\tau > 0$, considering inequalities

$$\|s_j^* a_{jj}\, s_j - a_{jj}\| < \varepsilon\,, \quad \text{and} \quad \|s_i^* a_{ij} s_j\| < \tau \ \text{ for } i \neq j\,. \tag{6}$$

Lemma 4.9. *Let $z_1, \ldots, z_n \in A$ such that $(z_1^* z_1, \ldots, z_n^* z_n)$ has the matrix diagonalization property in A.*

(i) *If $1 \leq k < n$, $e := z_1^* z_1 + \cdots + z_k^* z_k$ and $f := z_{k+1}^* z_{k+1} + \cdots + z_n^* z_n$, then (e, f) has the matrix diagonalization property.*

(ii) *The n-tuple $(z_1z_1^*, \ldots, z_nz_n^*)$ has the matrix diagonalization property.*

Proof. (i): Follows from [26, lem. 5.9].

(ii): Let $[a_{ij}] \in M_n(A)_+$ with $a_{jj} = z_jz_j^*$. By Remark 2.1(iv), $a_{ij} \in \overline{a_{ii}Aa_{jj}}$ and $a_{ij} = \lim_{k\to\infty} b_i^{(k)} a_{ij} b_j^{(k)}$ for the contractions $b_j^{(k)} := (a_{jj}+1/k)^{-1/2}a_{jj}^{1/2} \geq 0$. Consider the polar decompositions $z_j^* = v_j|z_j^*| = v_ja_{jj}^{1/2}$ of the z_j^* in A^{**}. Then $v_jb_j^{(k)} = z_j^*(a_{jj}+1/k)^{-1/2} \in A$, $v_jz_jz_j^* = z_j^*z_jv_j$, $z_j^*z_j = v_ja_{jj}v_j^*$ and $z_jz_j^* = v_j^*(z_j^*z_j)v_j$. It follows $v_ia_{ij}v_j^* \in A$ and $v_i^*v_ia_{ij}v_j^*v_j = a_{ij}$ for $i, j = 1, \ldots, n$. The diagonal matrix $V := \mathrm{diag}(v_1, \ldots, v_n)$ is a partial isometry in $M_n(A^{**})$, the matrix $[c_{ij}] := V[a_{ij}]V^*$ is a positive matrix in $M_n(A)$ with diagonal entries $c_{jj} = z_j^*z_j$, and $V^*[c_{ij}]V = [a_{ij}]$.

Let $\varepsilon \geq \tau > 0$. By assumption, there are $e_j \in A$ with

$$\|e_j^*z_j^*z_je_j - z_j^*z_j\| = \|e_j^*v_ja_{jj}v_j^*e_j - v_ja_{jj}v_j^*\| < \varepsilon/2 \quad \text{and} \quad \|e_i^*v_ia_{ij}v_j^*e_j\| < \tau/2$$

for $i, j = 1, \ldots, n$ and $i \neq j$. Let $\delta := \tau/(2 + 2(\max_j \|e_j\|)^2)$. We find $k \in \mathbb{N}$ such that $\|a_{ij} - f_ia_{ij}f_j\| < \delta$ for $f_j := b_j^{(k)}$. Let $s_j := (v_jf_j)^*e_j(v_jf_j)$. Since $v_jf_j = v_jb_j^{(k)} \in A$, we get that $s_j \in A$. This s_j satisfy $\| s_j^*a_{jj}s_j - a_{jj} \| < \delta\|e_j\|^2 + \varepsilon/2 + \delta \leq \varepsilon$ and $\|s_i^*a_{ij}s_j\| < \delta\|e_j\|^2 + \tau/2 \leq \tau$ giving (6). □

Lemma 4.10. *Let $a, b \in A_+$. Suppose that, for each $\varepsilon \in (0, \min(\|a\|, \|b\|)/4)$, there exist $e, f \in A_+$ and $d_1, d_2 \in A$ such that*

(i) $d_1^*ed_1 = (a - 3\varepsilon)_+$ *and* $d_2^*fd_2 = (b - 3\varepsilon)_+$,
(ii) $\varepsilon e = (a - (a - \varepsilon)_+)e$, $\varepsilon f = (b - (b - \varepsilon)_+)f$, *and*
(iii) (e, f) *has the matrix diagonalization property.*

Then (a, b) has the matrix diagonalization property.

Proof. Let $[a_{ij}] \in M_2(A)_+$ with $a_{11} := a$ and $a_{22} := b$, $\varepsilon > 0$ and $\tau > 0$. We show that there exists $v_1, v_2 \in A$ such that $v_1^*av_1 = (a-4\varepsilon)_+$, $v_2^*bv_2 = (b-4\varepsilon)_+$, and $\|v_1^*a_{12}v_2\| < \tau$. If $4\varepsilon \geq \min(\|a\|, \|b\|)$ let $v_1 := \lambda(a)$ and $v_2 := \lambda(b)$ with $\lambda(t) := t^{-1/2} \cdot (t - 4\varepsilon)_+^{1/2}$. If $\varepsilon < \min(\|a\|, \|b\|)/4$, let $e, f, d_1, d_2 \in A$ be elements with the properties in (i)–(iii). We define continuous functions ψ and φ on $[0, \infty)$ by $\psi(t) := \min(1, \varepsilon^{-1}t)$, $\varphi(t) := \varepsilon^{-1}t$ for $t \leq \varepsilon$ and $\varphi(t) := t^{-1}\varepsilon$ for $t > \varepsilon$. Notice that $\varphi(t)t = \varepsilon\psi(t)^2$.

Put $c_1 := \varepsilon^{-1}(a - (a - \varepsilon)_+) = \psi(a)$, $c_2 := \varepsilon^{-1}(b - (b - \varepsilon)_+) = \psi(b)$. Then $e = c_1e$, $f = c_2f$, $a\varphi(a) = \varepsilon c_1^2$ and $b\varphi(b) = \varepsilon c_2^2$. Since the elements are all positive, we get that e commutes with c_1 and that f commutes with c_2. It follows that $e = e^{1/2}c_1^2e^{1/2}$ and

$f = f^{1/2}c_2^2 f^{1/2}$. We let $g_1 := \varepsilon^{-1/2}\varphi(a)^{1/2}e^{1/2}$ and $g_2 := \varepsilon^{-1/2}\varphi(b)^{1/2}f^{1/2}$. The 2×2-matrix $[b_{ij}] := \mathrm{diag}(g_1,g_2)^*[a_{ij}]\mathrm{diag}(g_1,g_2)$ is positive and has entries $b_{11} = g_1^* a g_1 = e$, $b_{22} = g_2^* b g_2 = f$, and $b_{21}^* = b_{12} = g_1^* a_{12}\, g_2$.

Let $\gamma := \max\big(1\,,\, \|d_1\|^2\,,\, \|d_2\|^2\,\big)$ and $\delta := \min(\varepsilon,\tau)/\gamma$. The diagonalization property of (e,f) gives $S_1,S_2\in A$ with $\|S_1^* e S_1 - e\| < \delta$, $\|S_2^* f S_2 - f\| < \delta$, and $\|S_1^* b_{12}\, S_2\| < \delta$.

It follows that $T_j := S_j d_j$ $(j=1,2)$ satisfy

$$\|T_1^* e T_1 - (a-3\varepsilon)_+\| < \delta\gamma \le \varepsilon\,,\ \ \|T_2^* f T_2 - (b-3\varepsilon)_+\| < \varepsilon \ \text{ and } \ \|T_1^* b_{12} T_2\| < \delta\gamma \le \tau\,.$$

Thus, $h_j := g_j T_j$ satisfies $\|h_1^* a h_1 - (a-3\varepsilon)_+\| < \varepsilon$, $\|h_2^* b h_2 - (b-3\varepsilon)_+\| < \varepsilon$ and $\|h_1^* a_{12} h_2\| < \tau$. Use Remark 2.1(i) to get the desired $v_i := h_i q_i$ with suitable contractions $q_i \in A$. □

Proof of Theorem 1.1: Let $a,b\in A_+\backslash\{0\}$. We show that (a,b) has the matrix diagonalization property. This applies in particular to the positive matrix $[a^{1/2}, b^{1/2}]^\top[a^{1/2}, b^{1/2}] \in M_2(A)_+$ and proves that A is strongly purely infinite in the sense of Definition 3.1.

Let $\varepsilon \in (0, \min(\|a\|,\|b\|)/4)$, and $\gamma := (\|a\| - 3\varepsilon)^{-1}$. We show the existence of $e,f; d_1,d_2 \in A$ that satisfy the conditions (i)–(iii) of Lemma 4.10.

By Lemma 4.1(i), we find $y_1,\ldots,y_m,d_1 \in A$ satisfying $e\,(a-(a-\varepsilon)_+) = \varepsilon e$, $y_i(y_i)^*\in\mathcal{F}$ and $d_1^* e d_1 = (a-3\varepsilon)_+$ for $e := \sum_{i=1}^m y_i^* y_i$, because we can apply Lemma 4.1(i) to the elements $\gamma(a-3\varepsilon)_+$, $\varepsilon^{-1}((a-\varepsilon)_+ - (a-2\varepsilon)_+)$ and $\varepsilon^{-1}(a-(a-\varepsilon)_+)$ in place of the elements $a\le b\le c$ in 4.1(i).

In the same way one can see, that Lemma 4.1(i) gives elements $z_1,\ldots,z_n,d_2\in A$ such that $f\,(b-(b-\varepsilon)_+) = \varepsilon f$, $z_j(z_j)^*\in\mathcal{F}$ and $d_2^* f d_2 = (b-3\varepsilon)_+$ for $f := \sum_j z_j^* z_j$.

Since the sequence $(y_1y_1^*,\ldots,y_my_m^*, z_1z_1^*,\ldots,z_nz_n^*)$ has the matrix diagonalization property (by assumptions on $\mathcal{F}$) the Lemma 4.9 applies and shows that the sequences $(y_1^*y_1,\ldots,y_m^*y_m, z_1^*z_1,\ldots,z_n^*z_n)$ and (e,f) both have the matrix diagonalization property. Thus the elements $e,f\in A_+$ and $d_1,d_2\in A$ satisfy the conditions (i)–(iii) of Lemma 4.10, and (a,b) has the matrix diagonalization property by Lemma 4.10. □

5. Verification of the matrix diagonalization

Given subsets $\mathcal{F} \subseteq A_+$, $\mathcal{C} \subseteq A$ and $\mathcal{S} \subseteq A$. In this section we study questions related to the verification the matrix diagonalization property with respect to $\mathcal{S}$ and $\mathcal{C}$ for (finite) tuples of elements in $\mathcal{F}$. We study questions of the following type:

(Q1) *Under which conditions on $\mathcal{F}$, does it follow that $\mathcal{F}$ has the matrix diagonalization property?*

(Q2) *Under which conditions on $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{S}$ can the inequalities* (1) *be solved by some $s_1, s_2 \in \mathcal{S}$ for each given $(a_1, a_2, c, \varepsilon \geq \tau > 0)$ with $a_1, a_2 \in \mathcal{F}$, and $c \in \overline{\operatorname{span}(\mathcal{C})}$?*

An example of a possible condition for a positive answer to (Q1) is that $\mathcal{F}$ is invariant under *ε-cut-downs*, i.e., that for each $a \in \mathcal{F}$ and $\varepsilon \in (0, \|a\|)$ we have $(a - \varepsilon)_+ \in \mathcal{F}$ (cf. Lemma 5.4). The answer to the second question has to do with interplay of $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{S}$. This means that we have to require additional suitable conditions, e.g. that $\mathcal{S}^* \cdot \mathcal{C} \cdot \mathcal{S} \subseteq \mathcal{C}$.

We need this generalization because our applications are concerned with families $\mathcal{F}$ that are not invariant under *ε-cut-downs*, i.e., operations $a \mapsto (a - \varepsilon)_+$ for $a \in \mathcal{F}$ and $\varepsilon \in (0, \|a\|)$. An example is the proof of Theorem 1.3. It uses the following Lemma 5.2 that we could not directly deduce from [26]. We start by a definition allowing us to better control the matrix diagonalization property:

Definition 5.1. A n-tuple $(a_1, \ldots, a_n)$ of positive elements in A has *controlled* matrix diagonalization property with respect to $\mathcal{S}$ and $\mathcal{C}$ if there is a non-decreasing controlling function

$$(0, \infty) \ni t \mapsto D_n(t) := D_n(t; a_1, \ldots, a_n) \in [1, \infty)$$

such that for every $[a_{ij}] \in M_n(A)_+$ with $a_{jj} = a_j$ and $a_{ij} \in \mathcal{C}$ (for $i \neq j$) and $\varepsilon_j > 0, \tau > 0$ there are $s_1, \ldots, s_n \in \mathcal{S}$ that satisfy the inequalities (4) and have norms that satisfy

$$\|s_j\|^2 \leq D_n(1/\min(\varepsilon_1, \ldots, \varepsilon_n))\,.$$

If $\mathcal{S} = \mathcal{C} = A$ we say $(a_1, \ldots, a_n)$ has controlled matrix diagonalization (in A).

The following lemma in parts reduces the problem of considering arbitrary n-tuples to 2-tuples. We say "in parts" because the assumptions in Lemma 5.2 involve matrices in $M_2(A)$ that are not necessarily positive. This difficulty is solved in Lemma 5.3.

Lemma 5.2. *Let $\mathcal{F}$ be a subset of A_+. Suppose that for any given $a_1, a_2 \in \mathcal{F}$, there exists a non-decreasing function $t \mapsto D(t;\, a_1, a_2) < \infty$ such that for each $c \in a_1^{1/2} A\, a_2^{1/2}$*

and $\varepsilon \geq \tau > 0$ there exist $s_1, s_2 \in A$ that fulfill (1) and $\|s_j\|^2 \leq D(1/\varepsilon\,;\ a_1, a_2)$. Then any n-tuple of elements in $\mathcal{F}$ has the controlled matrix diagonalization in A.

Proof. We can suppose that all the functions $t \mapsto D(t;\, a_1, a_2) < \infty$ satisfy $D(t;\, a_1, a_2) \geq 1$ for all $t \in (0, \infty)$, upon replacing $D(t;\, a_1, a_2)$ by $\widetilde{D}(t;\, a_1, a_2) := \max(1, D(t;\, a_1, a_2))$.

Let $a_1, \ldots, a_{n+1} \in \mathcal{F}$ and let $[a_{jk}]$ be a positive matrix in $M_{n+1}(A)$ with diagonal entries $a_{jj} = a_j$.

We proceed by induction over $n \geq 1$, and prove each n-tuple of elements in $\mathcal{F}$ has the controlled matrix diagonalization. It suffice to prove the existence of a controlling function $t \mapsto D_{n+1}(t) = D_{n+1}(t; a_1, \ldots, a_n) < \infty$ with the property that, for every $\varepsilon \geq \tau > 0$, there exists $s_1, \ldots, s_{n+1} \in A$ that fulfill (6) and $\|s_j\|^2 \leq D_{n+1}(1/\varepsilon)$. (For general $\varepsilon_j > 0, \tau > 0$, set $\varepsilon := \min(\varepsilon_1, \ldots, \varepsilon_n)$ and decrease τ if $\tau > \varepsilon$.)

Base case $n = 1$: Let $\varepsilon \geq \tau > 0$ be given. We prove $D_2(t) := D(t; a_1, a_2)$ is a controlling function by finding $s_1, s_2 \in A$ fulfilling (6) and $\|s_j\|^2 \leq D_2(1/\varepsilon)$ for our choice of D_2. With $x := a_{12}$ the sequence $y_k := (a_1 + 1/k)^{-1/2} x (a_2 + 1/k)^{-1/2} \in A$ satisfies $\|y_k\| \leq 1$ and $x := \lim_k x_k$ for $x_k := a_1^{1/2} y_k a_2^{1/2}$, cf. Remark 2.1(iv). Let $\delta := \tau/(2 + 2D_2(1/\varepsilon))$. There is $k \in \mathbb{N}$ with $\|x - x_k\| < \delta$. By assumptions on D, there exist $s_1, s_2 \in A$, with $\|s_j\|^2 \leq D_2(1/\varepsilon)$, $\|s_j^* a_j\, s_j - a_j\| < \varepsilon$ and $\|s_1^* x_k s_2\| < \delta$. Then $\|s_1^* x s_2\| < \delta + D_2(1/\varepsilon)\|x - x_k\| < \tau$, giving (6).

We proceed by induction over $n \geq 2$. Suppose that each n-tuple $(h_1, \ldots, h_n)$ with $h_j \in \mathcal{F}$ has controlled matrix diagonalization with controlling functions $t \mapsto D_n(t;\, h_1, \ldots, h_n)$ having $h_1, \ldots, h_n$ as parameters. In particular, the functions $t \mapsto D_2(t; a_1, a_{n+1})$, $t \mapsto D_n(t; a_1, \ldots, a_n)$ and $t \mapsto D_n(t; a_2, \ldots, a_{n+1})$ used below could be different. We try to keep notations transparent by defining

$$D_2(t) := D_2(t; a_1, a_{n+1}), \quad D_n(t) := \max\{\, D_n(t; a_1, \ldots, a_n),\, D_n(t; a_2, \ldots, a_{n+1})\,\}.$$

Now let $\varepsilon \geq \tau > 0$ be given. We consider the following values

$$\varepsilon_2 := \varepsilon/3\,, \quad \varepsilon_1 := \varepsilon/(3D_n(3/\varepsilon)) \quad \text{and} \quad \varepsilon_0 := \varepsilon/(3D_n(1/\varepsilon_1))\,, \tag{7}$$

and choose $\tau_0,\, \tau_1,\, \tau_2 > 0$ such that

$$\tau_2 < \tau\,, \quad D_n(1/\varepsilon_2)\tau_1 < \tau\,, \quad D_n(1/\varepsilon_2)D_n(1/\varepsilon_1)\tau_0 < \tau\,.$$

Notice that $D_n(1/\varepsilon_1) = D_n(\, D_n(1/\varepsilon_2)/\varepsilon_2\,)$ and $\varepsilon/(3D_n(3/\varepsilon)) = \varepsilon_2/D_n(1/\varepsilon_2)$.

There are $d_1, d_{n+1} \in A$ with $\|d_j\|^2 \leq D_2(1/\varepsilon_0)$ such that $\|d_j^* a_j d_j - a_j\| < \varepsilon_0$ for $j = 1, n+1$ and $\|d_1^* a_{1,n+1} d_{n+1}\| < \tau_0$. We can use Remark 2.1(i) to modify d_1 and d_{n+1} suitably, such that that $d_j^* a_j d_j = (a_j - \mu)_+$ for some $\mu < \varepsilon_0$. Now consider the diagonal matrices $w_1 := \operatorname{diag}(a_1 - (a_1 - \mu)_+\,, 0, \ldots, 0,\, a_{n+1} - (a_{n+1} - \mu)_+\,)$ and $d := \operatorname{diag}(d_1, 1, \ldots, 1, d_{n+1})$ in $M_{n+1}(\mathcal{M}(A))$. The matrix $[b_{jk}] := w_1 + d^*[a_{jk}]d$ is positive in $M_{n+1}(A)$ with $b_{jj} = a_j$ and $b_{jk} = d_j^* a_{jk} d_k$ for $j \neq k$, and $\|b_{1,n+1}\| < \tau_0$.

By induction hypothesis and Remark 2.1(i), there exists a diagonal matrix $e = \operatorname{diag}(e_1, \ldots, e_n, 1)$ such that $\|e\|^2 \leq D_n(1/\varepsilon_1)$, $\|e_j^* b_{jk} e_k\| < \tau_1$ for $j \neq k \in \{1, \ldots, n\}$ and $e_j^* a_j e_j = (a_j - \nu)_+$ for $j = 1, \ldots, n$ and some $\nu < \varepsilon_1$.

Consider the diagonal matrix $w_2 := \operatorname{diag}(a_1 - (a_1 - \nu)_+, \ldots, a_n - (a_n - \nu)_+, 0)$ and the positive matrix $[c_{jk}] := e^*[b_{jk}]e + w_2$ with diagonal entries $c_{jj} = a_j$ and $\|c_{1,k}\| < \tau_1$ for $k = 2, \ldots, n$, and $\|c_{1,k}\| < D_n(1/\varepsilon_1)\tau_0$ for $k = n+1$.

Apply the induction assumption to the lower right $n \times n$ sub-matrix of $[c_{jk}]$, get a diagonal matrix $f := \operatorname{diag}(1, f_2, \ldots, f_{n+1})$ such that $\|f\|^2 < D_n(1/\varepsilon_2)$ and

$$\|f_j^* a_j f_j - a_j\| < \varepsilon_2\,, \qquad \|f_j^* c_{jk} f_j\| < \tau_2 \quad \text{for} \quad j \neq k \in \{2, \ldots, n+1\}\,.$$

The diagonal matrix $g := \operatorname{diag}(d_1 e_1, e_2 f_2, \ldots, e_n f_n, d_{n+1} f_{n+1})$ has norm

$$\|g\|^2 \leq \max\{D_2(1/\varepsilon_0)D_n(1/\varepsilon_1), D_n(1/\varepsilon_1)D_n(1/\varepsilon_2), D_2(1/\varepsilon_0)D_n(1/\varepsilon_2)\}$$

and satisfies

$$\|g_j^* a_{jk} g_k\| \leq \begin{cases} \tau_2, & \text{if } j \neq k \in \{2, \ldots, n+1\} \\ D_n(1/\varepsilon_2)D_n(1/\varepsilon_1)\tau_0, & \text{if } j = 1, k = n+1 \\ D_n(1/\varepsilon_2)\tau_1, & \text{if } j = 1, k = 2, \ldots, n \end{cases}$$

By assumption on τ_0, τ_1, τ_2 we get $\|g_j^* a_{jk} g_k\| < \tau$ for all $j \neq k \in \{1, \ldots, n+1\}$. Also

$$\|g_j^* a_j g_j - a_j\| < \max\{D_n(1/\varepsilon_1)\varepsilon_0 + \varepsilon_1\,,\ D_n(1/\varepsilon_2)\varepsilon_1 + \varepsilon_2\,,\ D_n(1/\varepsilon_2)\varepsilon_0 + \varepsilon_2\} \leq \varepsilon\,.$$

Thus, the $(n+1)$-tuple $(a_1, \ldots, a_n, a_{n+1})$ has the diagonalization property with (the clearly non-decreasing) controlling function $D_{n+1}(t; a_1, \ldots, a_n, a_{n+1}) := D_{n+1}(t)$ defined by

$$D_{n+1}(t) := \max\{D_2(1/\varepsilon_0)D_n(1/\varepsilon_1), D_n(1/\varepsilon_1)D_n(1/\varepsilon_2), D_2(1/\varepsilon_0)D_n(1/\varepsilon_2)\}$$

with $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ defined from $\varepsilon := 1/t$ as above in (7). □

Recall that $\mathrm{C}_c(0, \infty]_+$ denotes the set of all non-negative continuous functions φ on $[0, \infty)$ with $\varphi|[0, \eta] = 0$ for some $\eta \in (0, \infty)$, such that $\lim_{t \to \infty} \varphi(t)$ exists.

In the following Lemma 5.3 we show that approximate matrix diagonalization for positive matrices in $M_2(A)$ extends in suitable cases to approximate diagonalization of (certain) selfadjoint matrices. In particular notice that the solvability by $s_1, s_2 \in \mathcal{S}$ of the inequality (1) does not require the positivity of the selfadjoint 2×2-matrix $\left(\begin{smallmatrix} a_1 & c \\ c^* & a_2 \end{smallmatrix}\right)$ anymore.

Lemma 5.3. *Let $a_1, a_2 \in A_+$, $\varepsilon_0 > 0$ and non-empty subsets $\mathcal{C} \subseteq A$, $\mathcal{S} \subseteq A$ be given. Suppose that the following properties hold:*

- (i) *For every $\delta \in (0, \varepsilon_0)$, the pair $((a_1-\delta)_+, (a_2-\delta)_+)$ has the matrix diagonalization property with respect to $\mathcal{S}$ and $\mathcal{C}$ of Definition 4.6.*
- (ii) *$\varphi(a_1)c\varphi(a_2) \in \mathcal{C}$ for each $c \in \mathcal{C}$ and $\varphi \in \mathrm{C}_c(0, \infty]_+$.*
- (iii) *$\varphi(a_1)s, \varphi(a_2)s \in \mathcal{S}$ for each $s \in \mathcal{S}$ and $\varphi \in \mathrm{C}_c(0, \infty]_+$.*

Then, for each $c \in \mathcal{C}$, $\varepsilon \in (0, \varepsilon_0)$ and $\tau > 0$, there exist $s_1, s_2 \in \mathcal{S}$ that fulfill (1) and $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$.

Proof. Let $\varepsilon_0 \geq \varepsilon > 0$ and $\tau > 0$, and define $\gamma := \varepsilon/2$.

Condition (ii) on $\mathcal{C}$ implies $0 \in \mathcal{C}$. Therefore there are $d_1, d_2 \in \mathcal{S}$ that satisfy the inequalities $\|d_j^*(a_j - \gamma)_+ d_j - (a_j - \gamma)_+\| < \gamma$. Then $s_j := e_j d_j$ is in $\mathcal{S}$ and $e_j a_j e_j = (a_j - \gamma)_+$ for $e_j := \varphi(a_j)$, where $\varphi(t) := ((t-\gamma)_+/t)^{1/2}$, and the $\{a_j, 0, s_j, \varepsilon, \tau\}$ satisfy the general inequalities (1) with $c := 0$.

Since $\gamma\varphi(t)^2 \leq \varphi(t)^2 t$ we get $s_j^* s_j \leq \gamma^{-1} d_j^*(a_j - \gamma)_+ d_j$ and using the norm inequality, $d_j^*(a_j - \gamma)_+ d_j \leq \gamma + (a_j - \gamma)_+$. Since $\|\cdot\|$ is monotone on A_+, it follows $s_j = 0$ if $\gamma \geq \|a_j\|$, and $\|s_j\|^2 \leq \gamma^{-1}\|a_j\|$ if $\gamma < \|a_j\|$, so $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$.

Suppose that $\min(\|a_1\|, \|a_2\|) = 0$. If follows that the solution (s_1, s_2) for the above case with $c = 0$ gives $\min(\|s_1\|, \|s_2\|) = 0$. In particular (s_1, s_2) fulfill (1) and $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$ for each $c \in \mathcal{C}$ with $\|s_1^* c s_2\| = 0$.

It remains to check the case where $\min(\|a_1\|, \|a_2\|) > 0$. We let

$$\alpha_1 := \min(\|a_1\|, \|a_2\|) \quad \text{and} \quad \alpha_2 := \max(\|a_1\|, \|a_2\|)\,.$$

Let $c \in \mathcal{C}$. By decreasing τ if $\tau > \varepsilon$, we may assume $\tau \leq \varepsilon$.

We define functions $\chi, \psi, \varphi \in \mathrm{C}_c(0, \infty]_+$ by $\xi(t) := \min(\alpha_2, (t-\gamma)_+)^{1/2}$, $\chi(t) := \delta^{-1}\min((t-\delta)_+, \delta)$ with $\delta := \gamma/2$, $\psi(t) := \chi(t)t^{-1/2}$ and φ is as above defined. Notice that $\psi(t)^2(t-\gamma)_+ = \varphi(t)^2$, $\varphi(t)^2 t = (t-\gamma)_+$, $\xi(a_j) = (a_j - \gamma)_+^{1/2}$.

Let $e_j := \varphi(a_j)$ and $f_j := \psi(a_j)$. It follows that $e_j = (a_j - \gamma)_+^{1/2} f_j$ has norm $\|e_j\| = \|\varphi(a_j)\| \leq 1$. By assumption (iii), the elements $f_1 c f_2$ and $e_1 c e_2$ are in $\mathcal{C}$.

Case $f_1 c f_2 = 0$: Then $e_1 c e_2 = 0$. Since we do not know if e_j is in $\mathcal{S}$, we can not define s_j simply by $s_j := e_j$. But the above considered case $c = 0$ gives $d_1, d_2 \in \mathcal{S}$ with $\|d_j^*(a_j - \gamma)_+ d_j - (a_j - \gamma)_+\| < \gamma$. The $s_j := e_j d_j \in \mathcal{S}$ satisfy the inequalities (1) and $\|s_j\|^2 \leq \gamma^{-1}\|a_j\|$, where we use that $e_j a_j e_j = (a_j - \gamma)_+$ and above estimates for $\|s_j\|^2$.

Case $f_1 c f_2 \neq 0$: We define $\rho := \max(1, \|f_1 c f_2\|)^{-1}$ and $c' := \rho \cdot f_1 c f_2$ and put $\tau' := \rho\tau > 0$. Then $c' \in \mathcal{C}$ by assumption (ii), because $\sqrt{\rho}\psi \in \mathrm{C}_c(0,\infty]_+$, $\|c'\| \leq 1$ and the matrix $X = [b_{ij}] \in M_2(\mathcal{M}(A))$ with entries $b_{11} := b_{22} := 1$ and $b_{21}^* := b_{12} := c'$ is a positive matrix since $\|c'\| \leq 1$. Recall that $\xi(a_j) = (a_j - \gamma)_+^{1/2}$. Since $\xi(t)\psi(t) = \varphi(t)$ for $t \leq \alpha_2$, we have that $(a_j - \gamma)_+ = \xi(a_j) f_j a_j f_j \xi(a_j)$.

Let $\Delta := \mathrm{diag}((a_1-\gamma)_+^{1/2}, (a_2-\gamma)_+^{1/2})$. The 2×2-matrix $[y_{ij}] = Y := \Delta X \Delta \in M_2(A)_+$ has diagonal Δ^2 and the upper right element of Y is

$$y_{12} = \xi(a_1) c' \xi(a_2) = \rho (a_1 - \gamma)_+^{1/2} f_1 c f_2 (a_2 - \gamma)_+^{1/2}\,.$$

It is in $\mathcal{C}$ by condition (ii). By condition (i), $((a_1-\gamma)_+, (a_2-\gamma)_+)$ has the diagonalization property with respect to $\mathcal{S}$ and $\mathcal{C}$. Hence for each $\mu > 0$ there exist $d_1, d_2 \in \mathcal{S}$ such that the diagonal matrix $S = \mathrm{diag}(d_1, d_2)$ satisfies with respect to the norm of $M_2(A)$ the inequality

$$\|S^* Y S - \Delta^2\| < \mu\,.$$

E.g. we can take $0 < \mu < \min(\tau', \gamma, \|a_1\|, \|a_2\|)$. Then this implies

$$\rho \cdot \|d_1^* e_1\, c\, e_2 d_2\| = \|d_1^* (a_1 - \gamma)_+^{1/2}\, c'\, (a_2 - \gamma)_+^{1/2} d_2\| < \mu < \rho \cdot \tau$$

and

$$\|d_j^* e_j a_j e_j d_j - (a_j - \gamma)_+\| = \|d_j^*(a_j - \gamma)_+ d_j - (a_j - \gamma)_+\| < \mu\,.$$

The $s_j := e_j d_j$ $(j = 1, 2)$ are in $\mathcal{S}$ by assumption (iii) and fulfill the inequalities (1).

An upper estimate of the minimal possible norms of the $s_1, s_2 \in A$ that fulfill the inequalities (1) can now be deduced as above from

$$\gamma s_j^* s_j \leq d_j^*(a_j - \gamma)_+ d_j \leq \mu + (a_j - \gamma)_+\,.$$

It implies that $\gamma\|s_j\|^2 \leq \mu + (\|a_j\| - \gamma)_+ \leq \|a_j\|$. $\square$

Notice that the Lemma 5.3 and Lemma 5.2 together generalize [26, lem. 5.6,lem. 5.7].

Combining Lemma 5.2 and Lemma 5.3 we obtain the following result applicable to families $\mathcal{F} \subseteq A_+$ invariant under ε-cut-downs (if $\mathcal{F}$ is not invariant under ε-cut-downs one could enlarge $\mathcal{F}$):

Lemma 5.4. *Suppose that $\mathcal{F} \subseteq A_+$ is invariant under ε-cut-downs, i.e., that for each $a \in \mathcal{F}$ and $\varepsilon \in (0, \|a\|)$ we have $(a-\varepsilon)_+ \in \mathcal{F}$. Then the following conditions (i)–(iii) on $\mathcal{F}$ are equivalent:*

(i) *Each 2-tuple (a_1, a_2) with $a_1, a_2 \in \mathcal{F}$ has the matrix diagonalization property of Definition 4.6.*

(ii) *For every $(a_1, a_2) \in \mathcal{F} \times \mathcal{F}$, $c \in a_1^{1/2} A a_2^{1/2}$ and $\varepsilon > 0$, there exists $e_1, e_2 \in A$ such that $\|e_1^* c e_2\| < \varepsilon$ and $\|a_j - e_j^* a_j e_j\| < \varepsilon$ (for $j \in \{1, 2\}$).*

(iii) *For every $(a_1, a_2) \in \mathcal{F} \times \mathcal{F}$, $c \in A$ and $\varepsilon > 0$, there exists $d_1, d_2 \in A$ such that $\|d_1^* c d_2\| < \varepsilon$, $\|a_j - d_j^* a_j d_j\| < \varepsilon$ (for $j \in \{1, 2\}$), and $\|d_j\|^2 \leq 2\|a_j\|/\varepsilon$.*

In particular, the family $\mathcal{F}$ has the controlled matrix diagonalization property of Definition 5.1, if and only if, each pair of elements in $\mathcal{F}$ has the matrix diagonalization property of Definition 4.6.

Proof. Let $(a_1, a_2) \in \mathcal{F} \times \mathcal{F}$. Since $\big((a_1-\varepsilon_1)_+, (a_2-\varepsilon_2)_+\big) \in \mathcal{F} \times \mathcal{F}$ for $\varepsilon_j \in [0, \|a_j\|]$, we get that conditions (i) and (iii) are equivalent by [26, lem.5.6] or Lemma 5.3 with $\mathcal{C} := A$ and $\mathcal{S} := A$, which shows that the estimate $\|d_j\|^2 \leq \varepsilon^{-1} 2\|a_j\|$ follows from condition (i). Condition (iii) implies that the family $\mathcal{F}$ has the controlled matrix diagonalization property in the sense of Definition 5.1 by Lemma 5.2, because for $a_1, a_2 \in \mathcal{F}$ we can define a control $t \mapsto D_2(t; a_1, a_2)$ by $D_2(t; a_1, a_2) := 2\max(\|a_1\|, \|a_2\|) \cdot t$. Notice that $2\|a_j\|/\varepsilon \leq D_2(1/\varepsilon; a_1, a_2)$ for $j \in \{1, 2\}$. The controlled matrix diagonalization property of the family $\mathcal{F}$ with respect to $\mathcal{S} := \mathcal{C} := A$ follows from Lemma 5.2 if $\mathcal{F} \times \mathcal{F}$ satisfies condition (iii). Clearly condition (iii) implies condition (ii) on $\mathcal{F}$. It remains to show that condition (ii) implies condition (i) for $\mathcal{F} \times \mathcal{F}$.

(ii)$\Rightarrow$(i): By Remark 2.1(iv), the condition (ii) on the pairs $(a_1, a_2) \in \mathcal{F}$ is formally weaker than the matrix-decomposition property. But the fact that condition (ii) applies also to $\big((a_1-\delta)_+, (a_2-\delta)_+\big) \in \mathcal{F} \times \mathcal{F}$ – in place of (a_1, a_2) – allows to proof that (a_1, a_2) has the matrix diagonalization property, and that [26, lem. 5.6] (or alternatively Lemma 5.3 with $\mathcal{C} := (a_1-\delta)_+^{1/2} A (a_2-\delta)_+^{1/2}$ and $\mathcal{S} := A$ for each $\delta > 0$) applies and shows the estimate given in condition (iii), which is (i) and the estimate.

More generally, let $a_1, a_2 \in \mathcal{F} \subset A_+$. Then following observation on general $a_1, a_2 \in A_+$ applies to (a_1, a_2) and shows that (a_1, a_2) has the matrix diagonalization property

of Definition 4.6. It follows then that each of the conditions (i)–(iii) on $\mathcal{F} \times \mathcal{F}$ are equivalent.

Suppose that $a_1, a_2 \in A_+$ are non-zero and have the property that, for each $\delta, \tau \in (0, \min\{\|a_1\|, \|a_2\|\})$ and each $x \in A$, there exist $e_1, e_2 \in A$, depending on (x, δ, τ), with

$$\|e_j^*(a_j - \delta)_+ e_j - (a_j - \delta)_+\| < \tau \quad \textit{and} \quad \|e_1^*(a_1 - \delta)_+^{1/2} x (a_2 - \delta)_+^{1/2} e_2\| < \tau\,. \tag{8}$$

Then there exist for each $\varepsilon > 0$ and $c \in A$ elements $d_1, d_2 \in A$ with $\|d_j^* a_j d_j - a_j\| < \varepsilon$, $j \in \{1, 2\}$, and $\|d_1^* c d_2\| < \varepsilon$.

To verify this Observation, let $a_1, a_2 \in A_+$, $c \in A$ and $\varepsilon > 0$ given. Without loss of generality we can suppose that a_1, a_2 and c are non-zero and that $2\varepsilon \le \min(\|a_1\|, \|a_2\|)$.

Let $\tau := \gamma := \varepsilon/2$, and $\delta \in (0, \gamma)$. Then

$$\|(a_j - \delta)_+ - a_j\| \le \delta < \gamma$$

and there exists a bounded continuous real function $g_\delta \in \mathrm{C}_c(0, \infty]_+$ with $g_\delta(\xi) = 0$ on $[0, \delta/2]$ and $g_\delta(\xi) = \xi^{-1/2}$ for $\xi \ge \delta$. Let $h_\delta(\xi) := \xi^{-1/2}(\xi - \delta)_+^{1/2}$ for $\xi \in (0, \infty)$ and $h_\delta(0) := 0$.

Notice that $\xi^{1/2} h_\delta(\xi) = (\xi - \delta)_+^{1/2}$ and $g_\delta(\xi)(\xi - \delta)_+^{1/2} = h_\delta(\xi)$. Let $x := h_\delta(a_1) c h_\delta(a_2)$. By assumptions of the Observation, there exist $e_1, e_2 \in A$ with

$$\|e_j^*(a_j - \delta)_+ e_j - (a_j - \delta)_+\| < \tau \quad \text{and} \quad \|e_1^* x e_2\| < \tau\,.$$

Now let $d_j := h_\delta(a_j) e_j$. Then $e_1^* x e_2 = d_1^* c d_2$ and $d_j^* a_j d_j = e_j^*(a_j - \delta)_+ e_j$. Thus, $\|d_j^* a_j d_j - a_j\| \le \delta + \gamma < \varepsilon$ and $\|d_1^* c d_2\| \le \tau < \varepsilon$, as desired. □

In the following two Lemmata 5.5 and 5.6 we consider a globalization of Lemma 5.3 to the case of families $\mathcal{F} \subseteq A_+$ and the case where $\mathcal{S}$ is moreover a multiplicative sub-semigroup of A satisfying $s_2^* \mathcal{C} s_1 \subseteq \mathcal{C}$ for all $s_1, s_2 \in \mathcal{S}$. We introduce first a bit notation. Let $D\colon (0, \infty) \to [1, \infty)$ be any function and n any positive integer. For each $t \in (0, \infty)$ let $\mathcal{Y}_{t,n}$, or simply $\mathcal{Y}_t$, denote set[4] consisting of all n-tuples $(\varepsilon_1, \ldots, \varepsilon_n) \in (0, 1/t)^n$ that satisfy the inequality

$$\varepsilon_n + \sum_{k=1}^{n-1} \varepsilon_k \cdot D(1/\varepsilon_{k+1}) \cdot \ldots \cdot D(1/\varepsilon_n) \le 1/t\,. \tag{9}$$

Moreover, let $t \mapsto E_n(t) \in [1, \infty)$ denote the function defined by

$$E_n(t) := \inf\{\, D(1/\varepsilon_1) \cdot \ldots \cdot D(1/\varepsilon_n)\,;\; (\varepsilon_1\,, \ldots\,, \varepsilon_n) \in \mathcal{Y}_t\}\,.$$

[4] The set $\mathcal{Y}_t$ is nonempty for every $t > 0$ by Proof of Lemma 5.5.

Lemma 5.5. *Let $a_1, a_2 \in A_+$, $\varepsilon_0 \in (0, \infty]$ and non-empty subsets $\mathcal{S} \subseteq A$, $\mathcal{C} \subseteq A$ be given. Suppose that the following properties hold:*

(i) *The set $\mathcal{S}$ is a multiplicative sub-semigroup of A that satisfies $s_2^* \mathcal{C} s_1 \subseteq \mathcal{C}$ for all $s_1, s_2 \in \mathcal{S}$.*

(ii) *There exists a non-decreasing function $t \in (0, \infty) \mapsto D(t) := D(t; a_1, a_2) \in [1, \infty)$ such that for each $c \in \mathcal{C}$, each $\tau, \varepsilon \in (0, \varepsilon_0)$ with $\varepsilon \geq \tau$ there exist $s_1, s_2 \in \mathcal{S}$ that fulfill (1) and $\|s_j\|^2 \leq D(1/\varepsilon)$.*

Then one can find, for each finite subset $X \subseteq \mathcal{C}$ and $\varepsilon \geq \tau > 0$, elements $s_1, s_2 \in \mathcal{S}$ that satisfy (1) for every *$c \in X$. Moreover, if $\varepsilon \in (0, \varepsilon_0)$ and $n := |X|$, then we can ensure $\|s_j\|^2 \leq E_n(1/\varepsilon)$.*

For each $c \in A$ in the linear span of at most n elements $c_1, \ldots, c_n \in \mathcal{C}$ and each $\varepsilon \geq \tau > 0$, there exist $s_1, s_2 \in \mathcal{S}$ that fulfill (1). Moreover, if $\varepsilon \in (0, \varepsilon_0)$, then we can ensure $\|s_j\|^2 \leq E_n(1/\varepsilon)$.

If $D(t) \leq \gamma$ for a constant γ then $E_n(t) \leq \gamma^n$ for all $n \in \mathbb{N}$, and if $D(t) \leq \gamma \cdot t$ then an upper estimate for E_n is given by $E_n(t) \leq (nt\gamma)^{(2^n - 1)}$.

If $D(t) = 1$ then, for each c in the closure of the linear span of of $\mathcal{C}$, $\varepsilon \in (0, \varepsilon_0)$, and $\tau > 0$, there are contractions $s_1, s_2 \in \mathcal{S}$ that satisfy the inequalities (1).

Proof. If (a_1, a_2), $\mathcal{C}$, $\mathcal{S}$, ε_0 and $t \mapsto D(t)$ are given, then we can define for any positive $t \in \mathbb{R}$ numbers $\nu_0, \nu_1, \ldots, \nu_{n-1}$ by induction for $k = 0, 1, \ldots, n-1$ as follows: Let $\nu_0 := 2t$ and $\nu_{k+1} := 2D(\nu_k)\nu_k$. The n-tuple $(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_k := \nu_{n-k}^{-1}$ $(k = 1, \ldots n)$ satisfies the inequality (9) with "$<$" in place of "$\leq$". Thus $\mathcal{Y}_t$ is non-empty. An *alternative construction* is given by $\nu_0 := nt$ and $\nu_{k+1} := D(\nu_k)\nu_k$. Then the $\varepsilon_k := \nu_{n-k}^{-1}$ satisfy (9) with "$=$" in place of "$\leq$". We use the latter to find bounds for $E_n(t)$.

Given $X = \{x_1, \ldots, x_n\} \subseteq \mathcal{C}$ and $\varepsilon \geq \tau > 0$. If $\varepsilon \geq \varepsilon_0$ then we can decrease ε and τ below ε_0 in the following considerations and find the s_1, s_2 satisfying (1) for these smaller ε and τ. We only need to prove the norm estimate $\|s_j\|^2 \leq E_n(1/\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$. Therefore we can assume below always that $\varepsilon < \varepsilon_0$.

With $t := 1/\varepsilon$, $\mathcal{Y}_t$ is non-empty by the computation above. Let $(\varepsilon_1, \ldots, \varepsilon_n)$ an *arbitrary* element of $\mathcal{Y}_t$ and define τ_k from τ and the ε_k by

$$\tau_k := \min\{\tau / D(1/\varepsilon_{k+1}) \cdot \ldots \cdot D(1/\varepsilon_n), \varepsilon_k\}\,.$$

By assumptions applied to $\tau_k, \varepsilon_k \in (0, \varepsilon_0)$, we can find elements $s_1^{(k)}, s_2^{(k)} \in \mathcal{S}$ with norms $\|s_j^{(k)}\| \le D(1/\varepsilon_k)^{1/2}$ that satisfy the inequalities (for $j = 1, 2$, $k = 1, \ldots, n$)

$$\|(s_j^{(k)})^* a_j s_j^{(k)} - a_j\| < \varepsilon_k \quad \text{and} \quad \|(s_1^{(k)})^* \, c_k \, s_2^{(k)}\| < \tau_k \,, \tag{10}$$

where we let $c_1 := x_1$ and $c_{k+1} := \big(s_1^{(1)} \cdots s_1^{(k)}\big)^* x_{k+1} \big(s_2^{(1)} \cdots s_2^{(k)}\big) \in \mathcal{C}$.

The $s_j := s_j^{(1)} \cdots s_j^{(n)}$ $(j = 1, 2)$ satisfy

$$\|s_j\|^2 \le D(1/\varepsilon_1) \cdot \ldots \cdot D(1/\varepsilon_n) \tag{11}$$

and

$$\|s_1^* x_k s_2\| < \tau_k \cdot D(1/\varepsilon_{k+1}) \cdot \ldots \cdot D(1/\varepsilon_n) \le \tau \,.$$

Stepwise application of the triangle inequality and (10) shows that

$$\|s_j^* a_j s_j - a_j\| < \varepsilon_n + \sum_{k=1}^{n-1} \varepsilon_k D(1/\varepsilon_{k+1}) \cdot \ldots \cdot D(1/\varepsilon_n) \,.$$

Since $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{Y}_{1/\varepsilon}$ we get $\|s_j^* a_j s_j - a_j\| < \varepsilon$, ensuring (1) for each $c = x_k$. Since $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{Y}_{1/\varepsilon}$ was arbitrary, $\|s_j\|^2 \le E_n(1/\varepsilon)$.

The passage to the c in the linear span of finite subsets $X := \{x_1, \ldots, x_n\} \subseteq \mathcal{C}$ is a matter of finding a solution s_1, s_2 of the inequalities (1) for all $c \in X$ with appropriate choice of τ, (for $c = \sum_{i=1}^n \alpha_i x_i$ selecting s_j such that $\|s_1^* x_k s_2\| \le \tau/(1 + \sum_{i=1}^n |\alpha_i|)$ will do the job). The results on the norm estimates on s_1, s_2 remains unchanged.

The estimate $E_n(t) \le \gamma^n$ for constant $D(t) = \gamma$ follows easily by the definition of $E_n(t)$, because we know that $\mathcal{Y}_t$ is non-empty. If $D(t) = \gamma \cdot t$, $t \ge 1$, then take the ν_k as in the above alternative construction. It follows $(\frac{1}{\nu_{n-1}}, \ldots, \frac{1}{\nu_0}) \in \mathcal{Y}_t$, hence $E_n(t) \le D(\nu_0) \cdot \ldots \cdot D(\nu_{n-1}) = \nu_0 \cdot \ldots \cdot \nu_{n-1} \gamma^n$, and $\nu_k = (nt\gamma)^{2^k}/\gamma$. Thus $E_n(t) \le (nt\gamma)^{2^n - 1}$.

If $D(t) = 1$ then $E_n(t) = 1$ for all $n \in \mathbb{N}$. Hence we can decrease ε without enlarging $E(1/\varepsilon)$. Consequently, for c in the closed linear span of $\mathcal{C}$ and *any* $\varepsilon \in (0, \varepsilon_0)$, $\tau > 0$, we can find $c' \in \mathrm{span}(\mathcal{C})$ with $\|c' - c\| < \tau/2$ and contractions $s_1, s_2 \in \mathcal{S}$ that satisfy the inequalities (1) with $(c', \tau/2)$ in place of (c, τ). Then this s_1, s_2 also satisfy (1) with the given c and τ. □

Lemma 5.6. *Let $\varepsilon_0 > 0$ and non-empty subsets $\mathcal{F} \subseteq A_+$, $\mathcal{C} \subseteq A$ be given, and let $\mathcal{S} \subseteq A$ be a (multiplicative) sub-semigroup of A that satisfies $s_2^* \mathcal{C} s_1 \subseteq \mathcal{C}$ for all $s_1, s_2 \in \mathcal{S}$. Suppose that the following properties holds for every $a_1, a_2 \in \mathcal{F}$:*

(i) *For every $\delta \in (0, \varepsilon_0)$, the pair has $((a_1 - \delta)_+, (a_2 - \delta)_+)$ the matrix diagonalization property with respect to $\mathcal{S}$ and $\mathcal{C}$ of Definition 4.6.*

(ii) $\varphi(a_1)c\varphi(a_2) \in \mathcal{C}$ *for each* $c \in \mathcal{C}$ *and* $\varphi \in \mathrm{C}_c(0,\infty]_+$.

(iii) $\varphi(a_1)s, \varphi(a_2)s \in \mathcal{S}$ *for each* $s \in \mathcal{S}$ *and* $\varphi \in \mathrm{C}_c(0,\infty]_+$.

Then, for every $c \in \overline{\operatorname{span}(\mathcal{C})}$, $a_1, a_2 \in \mathcal{F}$, $\varepsilon \in (0, \varepsilon_0/2)$, *and* $\tau > 0$, *there exists* $s_1, s_2 \in \mathcal{S}$ *that fulfill* (1) *and* $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$.

Proof. Take any $a_1, a_2 \in \mathcal{F}$ and $\delta \in (0, \varepsilon_0/2)$. Due to property (i) the system $((a_j - \delta)_+, \varepsilon_0/2, \mathcal{C}, \mathcal{S})$ fulfills the conditions of Lemma 5.3. Hence, by Lemma 5.3, for every $c \in \mathcal{C}$, $\varepsilon \in (0, \varepsilon_0/2)$ and $\tau > 0$, there exist $s_1, s_2 \in \mathcal{S}$ with norms $\|s_j\|^2 \leq 2\|(a_j - \delta)_+\|/\varepsilon$ that satisfy the inequalities $\| \, s_j^*(a_j - \delta)_+ s_j - (a_j - \delta)_+ \, \| < \varepsilon$, and $\| \, s_1^* c s_2 \, \| < \tau$. Define $\varepsilon_1 := \varepsilon_0/2$, and $D_1(t) := \max(1, t \cdot 2 \max \|(a_j - \delta)_+\|)$. The system $((a_j - \delta)_+, \mathcal{C}, \mathcal{S})$ fulfills conditions of Lemma 5.5 using (ε_1, D_1) in place of (ε_0, D).

Hence, by Lemma 5.5, for every $c \in \operatorname{span}(\mathcal{C})$ and $\varepsilon \geq \tau > 0$, there exist $s_1, s_2 \in \mathcal{S}$ with $\| \, s_j^*(a_j - \delta)_+ s_j - (a_j - \delta)_+ \, \| < \varepsilon$, and $\| \, s_1^* c s_2 \, \| < \tau$. We obtain that the system $(a_j, \varepsilon_0/2, \operatorname{span}(\mathcal{C}), \mathcal{S})$ fulfills the conditions of Lemma 5.3. We can now apply Lemma 5.3 on arbitrary $a_1, a_2 \in \mathcal{F}$. It follows that for $a_1, a_2 \in \mathcal{F}$, $\varepsilon \in (0, \varepsilon_0/2)$, $\tau > 0$, and $c \in \operatorname{span}(\mathcal{C})$ there exist $s_1, s_2 \in \mathcal{S}$ with $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$ that satisfy the inequalities (1).

If $a_1, a_2 \in \mathcal{F}$, $\varepsilon \in (0, \varepsilon_0/2)$, and $\tau > 0$ are given and if $c = \lim_n c_n$ with $c_n \in \operatorname{span}(\mathcal{C})$, then we find $k \in \mathbb{N}$ with $\|c_k - c\| < \delta$, where $\delta := (\varepsilon \cdot \tau)/(2 + 4\sqrt{\|a_1\| \cdot \|a_2\|})$. We find $s_1, s_2 \in \mathcal{S}$ with $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$ that satisfy the inequalities (1) with $(c_k, \tau/2)$ in place of (c, τ). Then $\|s_1^* c s_2\| < \tau/2 + \delta\|s_1\|\|s_2\| \leq \tau$. Hence, for given $a_1, a_2 \in \mathcal{F}$, $c \in \overline{\operatorname{span}(\mathcal{C})}$, $\varepsilon \in (0, \varepsilon_0/2)$, and $\tau > 0$, there exist $s_1, s_2 \in \mathcal{S}$ that satisfy (1) and have norms $\|s_j\|^2 \leq 2\|a_j\|/\varepsilon$. □

6. Pure infiniteness of tensor products

The following Lemma 6.1 considers a subset $\mathcal{F} \subseteq A_+$ that is not invariant under ε-cut-downs. Therefore we use the definition of s.p.i. C^*-algebras that predicts that inequalities (3) can be solved by *contractions* s, t, see Remark 3.2. One could also work with weaker estimates for the $\|d_j\|$ that we can derive with our methods here, but that would require to prove first a more complicate version of the local Lemma 5.3 and then to update its globalization in Lemma 5.6. The key observation for such a generalization had to be started with comparison of ε-cut-downs with elements that actually exist in $\mathcal{F}$ and how these multiply $\mathcal{S}$ from the left and $\mathcal{C}$ form both sides. The

reader can find such a generalization for contractions $b \in B_+$, $c \in C_+$ and $0 \le \varepsilon \le 1$ e.g. with help of the inequalities $((b \otimes c) - 3\varepsilon)_+ \le (b-\varepsilon)_+ \otimes (c-\varepsilon)_+ \le ((b \otimes c) - \varepsilon^2)_+$.

Lemma 6.1. *Suppose that at least one of the C^*-algebras B or C is s.p.i. Then the family $\mathcal{F} := \{ b \otimes c;\ b \in B_+,\ c \in C_+ \}$ has the matrix diagonalization property in $B \otimes^\alpha C$ for each C^*-norm $\| \cdot \|_\alpha$ on the algebraic tensor product $B \odot C$.*

Proof. We consider the case where B is s.p.i. The case of s.p.i. C is similar.

If $b_1, b_2 \in B_+$, $f \in B$, $c_1, c_2 \in C_+$, $g \in C$, and $\varepsilon \ge \tau > 0$ are given, we let $\delta := \tau / \big(1 + \max\{ \|b_1\| + \|c_1\|, \|b_2\| + \|c_2\|, \|g\| \}\big)$. By Remark 3.2 there exists contractions $d_1, d_2 \in B$ such that $\|d_j^* b_j d_j - b_j\| < \delta$ and $\|d_1^* f d_2\| < \delta$. Since the set of positive contractions in C contains an approximative unit for C (cf. [33, thm. 1.4.2]), there exists a contraction $e \in C_+$ with $\|e c_j e - c_j\| < \delta$. The tensors $s_j := d_j \otimes e$ satisfy $\|s_j^*(b_j \otimes c_j)s_j - b_j \otimes c_j\| < \varepsilon$ and $\|s_1^*(f \otimes g)s_2\| < \tau$.

It follows that the pair (a_1, a_2) with $a_j := b_j \otimes c_j \in \mathcal{F}$, the subset $\mathcal{C} := \{ b \otimes c;\ b \in B,\ c \in C \}$ of $A := B \otimes^\alpha C$ and the multiplicative sub-semigroup $\mathcal{S} := \{ s \otimes e;\ s \in B,\ e \in C,\ \|s\| \le 1,\ \|e\| \le 1 \}$ of the algebraic tensor product $B \odot C \subseteq B \otimes^\alpha C$ satisfy the assumptions of Lemma 5.5 with $\varepsilon_0 := +\infty$. Since $\mathcal{S}$ consists of contractions, the corresponding estimating function is $D(t) = 1$. The closed linear span of $\mathcal{C}$ is dense in $B \otimes^\alpha C$. Now Lemma 5.5 gives that for each $a_1, a_2 \in \mathcal{F}$, $c \in B \otimes^\alpha C$, $\varepsilon > 0$ and $\tau > 0$ there exist contractions $s_1, s_2 \in \mathcal{S}$ that satisfy the inequalities (1). Therefore, Lemma 5.2 applies to $\mathcal{F}$ and we obtain that $\mathcal{F}$ has the diagonalization property in A. □

Lemma 6.2. *It B and C are C^*-algebras where B or C is exact, then the subset $\mathcal{F} = \{ b \otimes c;\ b \in B_+, c \in C_+ \}$ of $(B \otimes^{\min} C)_+$ is a filling family for $B \otimes^{\min} C$.*

Proof. Suppose that one of the algebras B or C is an exact C^*-algebra, that D is a hereditary C^*-subalgebra of $B \otimes^{\min} C$ and that I a primitive ideal of $B \otimes^{\min} C$ with $D \not\subseteq I$. Then [7, prop. 2.16(iii)], [7, prop. 2.17(ii)] and [7, lem. 2.18] together show that there exist non-zero $g \in B_+$, $h \in C_+$, $t \in B \otimes^{\min} C$ and pure states φ on B and ψ on C such that $(\varphi \otimes \psi)(I) = \{0\}$, $tt^* \in D$, $t^* t = g \otimes h$, $\varphi(g) = \|g\| = 1$ and $\psi(h) = \|h\| = 1$.

Thus, the subset $\mathcal{F} = \{ b \otimes c;\ b \in B_+, c \in C_+ \} \subseteq (B \otimes^{\min} C)_+$ satisfies the property (ii) of Lemma 4.1 for all primitive ideals I of $B \otimes^{\min} C$ with $D \not\subseteq I$. By Definition 4.2 the set $\mathcal{F}$ is a filling family for $B \otimes^{\min} C$. □

Proof of Theorem 1.3: The subset $\mathcal{F} := \{ a \otimes b;\ a \in A_+, b \in B_+ \} \subseteq (A \otimes^{\min} B)_+$ is a filling family for $A \otimes^{\min} B$ by Lemma 6.2, and has the matrix diagonalization

property in $A \otimes^{\min} B$ by Lemma 6.1. It follows that $A \otimes^{\min} B$ is s.p.i. by Theorem 1.1. $\square$

Example 6.3. The statement of Theorem 1.3 does not hold for the maximal tensor product of C^*-algebras: Let $A := R \otimes^{\max} C^*_\lambda(F_2)$, $B := C^*_\lambda(F_2)$, where R denotes the stably infinite simple unital nuclear C^*-algebra with finite unit element constructed by M. Rørdam [36], and F_2 is the free group on two generators. The algebras A and B are exact, and A is s.p.i. by [7, cor. 3.11]. The maximal C^*-tensor product $A \otimes^{\max} B$ is even not locally purely infinite (cf. [7] for a definition), because $R \otimes^{\max} \mathcal{K}$ is an ideal of a quotient of $A \otimes^{\max} B$. This follows from the fact that the C^*-algebra generated by the "two-sided" regular representation $(g,h) \mapsto \lambda(g)\rho(h)$ of $F_2 \times F_2$ on $\ell_2(F_2)$ contains the compact operators in its closed linear span, cf. [1].

7. Endomorphism crossed product

Let $\varphi\colon A \to A$ be a *-endomorphism of a C^*-algebra A that is *not necessarily injective*. We let $A_\infty := \ell_\infty(A)/c_0(A)$ and denote by $(A_e, \mathbb{Z}, \sigma)$ the canonical C^*-dynamical system associated with φ.

More precisely, we consider the inductive limit $(A_e, \varphi_m\colon A \to A_e)$ in the *category* of C^*-algebras of the sequence $A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$. The natural realization of A_e is the closure of the set $\bigcup_m \varphi_m(A)$ in A_∞, where $\varphi_1(a) := (a, \varphi(a), \varphi^2(a), \ldots) + c_0(A)$ and where $\varphi_n(a) := S_+^{n-1}(a, \varphi(a), \varphi^2(a), \ldots) + c_0(A)$ for the forward shift $S_+(a_1, a_2, \ldots) := (0, a_1, a_2, \ldots)$ on $\ell_\infty(A)$.

The backward shift $\sigma\colon (a_1, a_2, \ldots) + c_0(A) \mapsto (a_2, a_3, \ldots) + c_0(A)$ is an automorphism of $A_\infty = \ell_\infty(A)/c_0(A)$ that induces an automorphism $\sigma|A_e\colon A_e \to A_e$ because $\varphi_n = \sigma \circ \varphi_{n+1}$ and A_e is the closure of the union of $\varphi_1(A) \subseteq \varphi_2(A) \subseteq \ldots$. We denote the restriction $\sigma|A_e$ of σ to A_e simply again by σ. The corresponding $\mathbb{Z}$-action given by $n \mapsto \sigma^n$ will be also denoted by σ and is usually called *the action of the integers $\mathbb{Z}$ on A_e corresponding to φ.*

The *-homomorphisms σ, φ_n and φ satisfy the equations

$$\sigma \circ \varphi_n = \varphi^\infty \circ \varphi_n \quad , \quad \varphi_n = \varphi_\ell \circ \varphi^{\ell-n} = \sigma^{\ell-n} \circ \varphi_\ell \quad \text{and} \quad \sigma^k \circ \varphi_n = \varphi_n \circ \varphi^k ,$$

where $1 \le n \le \ell$, and where $\varphi^\infty((a_1, a_2, \ldots) + c_0(A)) := (\varphi(a_1), \varphi(a_2), \ldots) + c_0(A)$.

Another explanation for these formulas can be seen from the formulas given by Cuntz in [10, p. 101] for the restriction of φ^∞ to A_e by the commuting diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\varphi} & A & \xrightarrow{\varphi} & A & \xrightarrow{\varphi} & \cdots \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \\ A & \xrightarrow{\varphi} & A & \xrightarrow{\varphi} & A & \xrightarrow{\varphi} & \cdots \end{array} .$$

Recall that $\mathcal{M}(A)$ denotes the (two-sided) multiplier algebra of A. Consider the following non-degeneracy property (ND) and corner property (CP):

(ND) $A \neq I_\varphi$ for $I_\varphi := \overline{\bigcup_n (\varphi^n)^{-1}(0)}$.

(CP) The hereditary C^*-subalgebra $\varphi(A)A\varphi(A)$ of A is a corner of A. Equivalently: φ extends to a strictly continuous *-homomorphism $\mathcal{M}(\varphi)\colon \mathcal{M}(A) \to \mathcal{M}(A)$.

Lemma 7.1. *Let $\varphi\colon A \to \mathcal{M}(B)$ a *-homomorphism and consider the hereditary C^*-subalgebra $D := \varphi(A)B\varphi(A)$ of B. The following are equivalent:*

(i) *φ extends to a strictly continuous *-homomorphism $\mathcal{M}(\varphi)\colon \mathcal{M}(A) \to \mathcal{M}(B)$.*

(ii) *D is a corner of B, i.e., there is a projection $p \in \mathcal{M}(B)$ with $pBp = D$.*

(iii) *The (two-sided) annihilator* $\mathrm{Ann}(D,B) := \{b \in B\,;\, bD = \{0\} = Db\}$ *of D in B has the property that the C^*-subalgebra $D + \mathrm{Ann}(D,B)$ of B contains an approximate unit of B.*

Proof. It is well know and easy to see that if $C \subseteq \mathcal{M}(B)$ is a C^*-subalgebra such that $CBC = B$, then there is a unique unital C^*-morphism ψ from $\mathcal{M}(C)$ into $\mathcal{M}(B)$ that extends the inclusion map $C \hookrightarrow \mathcal{M}(B)$ (i.e., $\psi(c) = c$ for $c \in C$) and is strictly continuous with respect to the strict topologies on $\mathcal{M}(C)$ and $\mathcal{M}(B)$.

(i)$\Rightarrow$(ii,iii): Since $\mathcal{M}(\varphi)$ is strictly continuous, we get for $p := \mathcal{M}(\varphi)(1_{\mathcal{M}(A)})$ that $D = pBp$, and $pBp + (1-p)B(1-p)$ contains an approximate unit, e.g. $pe_\tau p + (1-p)e_\tau(1-p)$ if e_τ is an approximately central with p commuting with the approximate unit of B.

(iii)$\Rightarrow$(ii): Suppose that $E := D + \mathrm{Ann}(D,B)$ contains an approximate unit of B, then $EBE = B$. Thus there exist $\psi\colon \mathcal{M}(E) \to \mathcal{M}(B)$ unital and strictly continuous, with $\psi(c) = c$ for $c \in E$. Since $D \cdot \mathrm{Ann}(D,B) = \{0\}$ we get that $E = D + \mathrm{Ann}(D,B)$ is naturally isomorphic to $D \oplus \mathrm{Ann}(D,B)$. Thus $\mathcal{M}(E) \cong \mathcal{M}(D) \oplus \mathcal{M}(\mathrm{Ann}(D,B))$ and $p := \psi(1_{\mathcal{M}(A)})$ is a projection in B with $pdp = \psi(d)$ for $d \in D$. If $b = pbp$, then $b = \lim d_\tau b d_\tau$ for an approximate unit (d_τ) of D. Thus $pBp = D$.

(ii)$\Rightarrow$(i): If $p \in \mathcal{M}(B)$ is a projection satisfying $pBp = D$ then $\mathcal{M}(D) \cong p\mathcal{M}(B)p$ by a natural isomorphism that satisfies $\eta(T)pbp = Tpbp$ for $T \in \mathcal{M}(D) = \mathcal{M}(pBp)$

and is strictly continuous a map from $\mathcal{M}(D)$ to $\mathcal{M}(B)$. Since $D = \varphi(A)B\varphi(A)$, we can define a C^*-morphism λ from A into $\mathcal{M}(D)$ by $\lambda(a)d := \varphi(a)d$ for $a \in A$ and $d \in D$. The C^*-morphism λ is non-degenerate because $A \cdot A = A$ implies that $\lambda(A)D = \varphi(A)D = D$ by definition of D. We have seen that non-degenerate λ extends to a strictly continuous unital C^*-morphism $\mathcal{M}(\lambda)$ from A into $\mathcal{M}(D)$ (in a unique way). Define $\mathcal{M}(\psi)\colon \mathcal{M}(A) \to \mathcal{M}(B)$ by $\mathcal{M}(\varphi)(T) := \eta(\mathcal{M}(\lambda)(T)) \in p\mathcal{M}(B)p \subseteq \mathcal{M}(B)$. We have that $\mathcal{M}(\varphi)$ is strictly continuous because η and $\mathcal{M}(\lambda)$ are strictly continuous. □

In particular, the non-degeneracy condition "$\varphi(A)B = B$" is sufficient for the existence of $\mathcal{M}(\varphi)$.

Lemma 7.2. *If $\varphi\colon A \to A$ satisfies the above defined property (CP) then $\varphi_1\colon A \to A_e$ extends to a strictly continuous *-homomorphism $\mathcal{M}(\varphi_1)\colon \mathcal{M}(A) \to \mathcal{M}(A_e)$.*

Proof. Let $I := I_\varphi \subseteq A$ denote the closure of $\bigcup_n (\varphi^n)^{-1}(0)$ and suppose that $I \neq A$, i.e., $A_e \neq \{0\}$. Then $\varphi(I) = I$ and the related class map $\psi := [\varphi]_I$ defines an injective endomorphism of A/I.

Notice that epimorphisms $\pi_I\colon A \to A/I$ always extend to strictly continuous *-homomorphisms $\mathcal{M}(\pi_I)\colon \mathcal{M}(A) \to \mathcal{M}(A/I)$.

The *-homomorphism $\varphi\colon A \to A$ extends to a strictly continuous *-homomorphism $\mathcal{M}(\varphi)\colon \mathcal{M}(A) \to \mathcal{M}(A)$ if and only if $B := \varphi(A) + \mathrm{Ann}(\varphi(A))$ contains an approximate unit of A by the above mentioned argument.

The natural image $\pi_I(B)$ of $B = \varphi(A) + \mathrm{Ann}(\varphi(A))$ in A/I contains an approximate unit of A/I and $\pi_I(B) \subseteq C + \mathrm{Ann}(C)$ for $C := \pi_I(\varphi(A)) = \psi(A/I)$. It follows that $\pi_I \circ \varphi\colon A \to A/I$ extends to strictly continuous *-homomorphism $\mathcal{M}(\pi_I \circ \varphi)$ from $\mathcal{M}(A)$ into $\mathcal{M}(A/I)$ and that $\psi\colon A/I \to A/I$ extends to a strictly continuous *-monomorphism $\mathcal{M}(\psi)$ from $\mathcal{M}(A/I)$ into $\mathcal{M}(A/I)$. If we consider any of the *-homomorphisms $\varphi_n\colon A \to A_e$ then they factorize over A/I and there is a natural isomorphism θ from A_e onto $(A/I)_e$, where $(A/I)_e$ denotes the inductive limit for the endomorphism $\psi\colon A/I \to A/I$. It has the following transformations: $\theta \circ \varphi_n = \psi_n \circ \pi_I$, $\psi_{n+1} \circ \psi = \psi_n$ and $\psi_n = \sigma \circ \psi_{n+1}$, where ψ_n denotes the canonical map $\psi_{n,\infty}\colon A/I \to A_e$ from the n'th copy of A/I into the inductive $A_e \subseteq A_\infty = \ell_\infty(A)/c_0(A)$.

There is a sequence (constructed below) of C^*-subalgebras $D_1, D_2, \ldots$ of $(A/I)_e$ such that $D_n \subseteq \psi_{n+1}(A/I)$, $D_n\psi_n(A/I) = \{0\}$, and such that the vector space sum $\psi_1(A/I) + D_1 + D_2 + \cdots$ contains an approximate unit of $(A/I)_e$.

Since $D_n D_m = \{0\}$ for $n \neq m$ and $D_n \psi_1(A/I) = \{0\}$, this shows that $\psi_1(A/I) + \mathrm{Ann}(\psi_1(A/I), (A/I)_e)$ contains an approximate unit of $(A/I)_e$, and we can conclude that $\psi_1 \colon A/I \to (A/I)_e$ extends to a strictly continuous *-monomorphism $\mathcal{M}(\psi_1)$ from $\mathcal{M}(A/I)$ into $\mathcal{M}((A/I)_e)$. Since θ is an isomorphism from A_e onto $(A/I)_e$ and $\varphi_1 = \theta^{-1} \circ \psi_1 \circ \pi_I$, we get that the the superposition of strictly continuous *-homomorphisms $\mathcal{M}(\theta)^{-1} \circ \mathcal{M}(\psi_1) \circ \mathcal{M}(\pi_I)$ gives a strictly continuous *-homomorphism from $\mathcal{M}(A)$ into $\mathcal{M}(A_e)$ that extends $\varphi_1 \colon A \to A_e$.

The algebras $D_n \subseteq \psi_{n+1}(A/I)$ can be defined by $D_n := \psi_{n+1}(\mathrm{Ann}(\psi(A/I), A/I))$ using that $\psi_{n+1} \circ \psi = \psi_n$. Then inductively $\psi_1(A/I) + D_1 + \cdots + D_n$ contains an approximate unit of $\psi_{n+1}(A/I)$, which implies the stipulated existence of an approximate unit for $(A/I)_e$ in $\psi_1(A/I) + D_1 + D_2 + \cdots$. $\square$

There are non-equivalent definitions of crossed products by an endomorphism in the literature that lead to non-isomorphic crossed product C^*-algebras. Since there are different definitions of endomorphism crossed products $A \rtimes_\varphi \mathbb{N}$ of A by the additive semigroup $\mathbb{N}$ of natural numbers, we describe our definition and notation that is inspired by the definitions given by J. Cuntz [9], [10], W.L. Paschke [32] and P.J. Stacey [41]. See [18] and [21], for a general descriptions of such constructions and alternative definitions that give different crossed products by $\mathbb{N}$.

The C^*-algebra crossed product $A \rtimes_\varphi \mathbb{N}$ associated to an endomorphism φ of A was defined by J. Cuntz in [10, p. 101] for the special case where A is unital but φ is not necessarily unital. It was inspired by his special construction in [9] that showed that $\mathcal{O}_n$ is a semi-crossed product of M_{n^∞} by the endomorphism $\varphi(a) := e \otimes a$ for $e := \mathrm{diag}(1, 0, \ldots, 0)$. Since then there where several attempts to generalize his construction, but not necessarily in a way that is suitable for our applications.

The generalization of Stacey [41] suffers from his assumption that for *each* *-endomorphism $\varphi \colon A \to A$ the natural morphism $\varphi_1 \colon A \to A_e$ extends to a *-homomorphism from $\mathcal{M}(A)$ into $\mathcal{M}(A_e)$, or at least to a *-homomorphism from $\mathcal{M}(A)$ into $\mathcal{M}(A_e \rtimes_\sigma \mathbb{Z})$. But this is not the case for general injective *-endomophisms φ of A, even if φ satisfies the above non-degeneracy condition (ND), cf. Example 8.5.

With help of the above Lemmata 7.1–7.2 we extend the definition of J. Cuntz's [10, p. 101] to the non-unital case as follows:

Definition 7.3. Let φ be en endomorphism of a C^*-algebra A that satisfy the non-degeneracy property (ND) and the corner property (CP). We define $A \rtimes_\varphi \mathbb{N}$ to be the hereditary C^*-subalgebra of $A_e \rtimes_\sigma \mathbb{Z}$ that is generated by the image $\varphi_1(A)$ of A.

Our endomorphism φ of A should satisfy the above discussed non-degeneracy property (ND) and the corner property (CP). Indeed, let $B := A \rtimes_\varphi \mathbb{N}$ denote the crossed product induced by an endomorphism φ on A as defined by J. Cuntz in [10, p. 101]. Knowing that (ND) and (CP) holds one can *formally* define B as the hereditary C^*-subalgebra of $A_e \rtimes_\sigma \mathbb{Z}$ that is generated by the image $\varphi_1(A)$ of A. It is even a full hereditary C^*-subalgebra of $A_e \rtimes_\sigma \mathbb{Z}$, because $A_e \rtimes_\sigma \mathbb{Z}$ is generated by $u^n \varphi_1(A) u^m$ for $n, m \in \mathbb{Z}$.

It seems not always to be the case that Stacey's version of crossed product $A \rtimes_\varphi \mathbb{N}$ (see [41, def. 3.1] for $A \rtimes_\varphi^1 \mathbb{N}$) is naturally isomorphic to the hereditary C^*-subalgebra of $A_e \rtimes_\sigma \mathbb{Z}$ generated by $\varphi_1(A)$. However it is the case when $\varphi\colon A \to A$ extends to a strictly continuous *-homomorphism $\mathcal{M}(\varphi)\colon \mathcal{M}(A) \to \mathcal{M}(A)$ – or equivalently – that $\varphi_1\colon A \to A_e$ extend to $\mathcal{M}(\varphi_1)\colon \mathcal{M}(A) \to \mathcal{M}(A_e)$, cf. Lemma 7.2. Then, with $p := \mathcal{M}(\varphi_1)(1_{\mathcal{M}(A)})$, we have that $p(A_e \rtimes_\sigma \mathbb{Z})p$ *is the same* as the hereditary C^*-subalgebra of $A_e \rtimes_\sigma \mathbb{Z}$ generated by $\varphi_1(A)$ and is naturally isomorphic to Stacey's $A \rtimes_\varphi \mathbb{N}$.

Let us remind the reader of the definition of a G-separating action which applies to Proposition 7.7, and hence also Remark 7.5 and Lemma 7.6.

Definition 7.4 ([28, def. 5.1])**.** Suppose that (A, G, σ) is a C^*-dynamical system with discrete group G. The action of G on A is *G-separating* if for each $a_1, a_2 \in A_+$, $c \in A$ and $\varepsilon > 0$, there exist $d_1, d_2 \in A$ and $g_1, g_2 \in G$ such that

$$\|d_j^* a_j d_j - \sigma_{g_j}(a_j)\| < \varepsilon \quad \text{and} \quad \|d_1^* c d_2\| < \varepsilon \tag{12}$$

Remark 7.5 ([28, rem. 5.4])**.** Let (A, G, σ) a C^*-dynamical system.

(i) For each $a_1, a_2 \in A_+$, $x, d_1, d_2 \in A$, $g_0, g_1, g_2 \in G$ and $s_1 := d_1 U(g_1)$, $s_2 := \sigma_{g_0^{-1}}(d_2) U(g_0^{-1} g_2 g_2)$, $c := xU(g_0)$, $b_1 := a_1$, and $b_2 := \sigma_{g_0}(a_2)$ the following equalities hold:

$$\|s_j^* a_j s_j - a_j\| = \|d_j^* b_j d_j - \sigma_{g_j}(b_j)\| \quad \text{and} \quad \|s_1^* c s_2\| = \|d_1^* x d_2\| \, .$$

(ii) With $g_0 = e$ in (i) the equalities reduce to:

$$\|s_j^* a_j s_j - a_j\| = \|d_j^* a_j d_j - \sigma_{g_j}(a_j)\| \quad \text{and} \quad \|s_1^* c s_2\| = \|d_1^* c d_2\| \, .$$

Lemma 7.6. *Let (A, G, σ) a C^*-dynamical system. The following properties are equivalent:*

(i) *The action σ is G-separating, i.e., for each $a_1, a_2 \in A_+$, $c \in A$ and $\varepsilon > 0$, there exist $d_1, d_2 \in A$ and $g_1, g_2 \in G$ satisfying (12).*

(ii) *There exists a dense *-subalgebra $\mathcal{B}$ of A that has the properties (1)–(3):*
  (1) *$\psi(b^*b) \in \mathcal{B}$ for all $\psi \in \mathrm{C}_c(0, \infty]_+$ and $b \in \mathcal{B}$.*
  (2) *$\sigma_g(\mathcal{B}) \subseteq \mathcal{B}$ for all $g \in G$.*
  (3) *For each $b_1, b_2 \in \mathcal{B}$, $\varepsilon > 0$ and $c \in \mathcal{B}$ there exist $d_1, d_2 \in A$ and $g_1, g_2 \in G$ that satisfy the inequalities (12) with $a_j := b_j^* b_j$ ($j = 1, 2$).*

Proof. Clearly (i)⇒(ii) with $\mathcal{B} := A$.

(ii)⇒(i): Define $\mathcal{C} := \{dU(g)\,;\ d \in \mathcal{B},\ g \in G\}$ and $\mathcal{S} := \mathcal{C}$. We use Lemma 5.6 on the above defined $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{F} := \{b^*b :\ b \in \mathcal{B}\}$. The property that $s_2^* \mathcal{C} s_1 \subseteq \mathcal{C}$ for all $s_1, s_2 \in \mathcal{S}$ and conditions (ii)-(iii) of Lemma 5.6 are trivially satisfied using (ii)(1)-(ii)(2). Moreover, by (ii)(1), the family $\mathcal{F}$ is invariant under ε-cut-downs, because $(b^*b - \varepsilon)_+ = \psi(b^*b)^* \psi(b^*b)$ for the function $\psi(t) := \min\big((t - \varepsilon)_+, \|b\|^2\big)^{1/2} \in \mathrm{C}_c(0, \infty]_+$. This implies that also condition (i) of Lemma 5.6 is fulfilled by $\mathcal{F}$, because $\mathcal{B}$ satisfies condition (ii)(3): Take any $\varepsilon_0 > 0$, and $a_j = b_j^* b_j \in \mathcal{F}$ for $j = 1, 2$. For each $c = xU(g_0) \in \mathcal{C}$ with $x \in \mathcal{B}, g_0 \in G$, and $0 < \tau \leq \varepsilon \leq \varepsilon_0$ we can use (ii)(3) to find elements $d_1, d_2 \in \mathcal{B}$ and $g_1, g_2 \in G$ satisfying (12) with $x, \sigma_{g_0}(a_2), \tau$ in place of c, a_2, ε. Remark 7.5 provides elements $s_1, s_2 \in \mathcal{S}$ satisfying (1). So the pair (a_1, a_2) has the matrix diagonalization with respect to $\mathcal{S}$ and $\mathcal{C}$ and property (i) of Lemma 5.6 holds.

We obtain from Lemma 5.6 that for every $c \in A$, $a_1, a_2 \in \mathcal{B}$, $\varepsilon_0/2 \geq \varepsilon > 0$, and $\tau > 0$, there exist $s_1, s_2 \in \mathcal{S}$ satisfying (1). Using Remark 7.5 we can find (for given a_j, c, ε) elements $d_1, d_2 \in A$ and $g_1, g_2 \in G$ satisfying (12).

□

In the following proposition we consider a dense *-subalgebra $\mathcal{B} \subseteq A$ that is φ-*invariant* – in the sense that $\varphi(\mathcal{B}) \subseteq \mathcal{B}$ – and $\mathcal{B}$ is a C^*-*local* subalgebra – in the sense that $\psi(b^*b) \in \mathcal{B}$ for $b \in \mathcal{B}$ and $\psi \in \mathrm{C}_c(0, \infty]_+$ (see definition in Section 2). For example, $\mathcal{B}$ can be an algebraic inductive limit of an upward directed family of C^*-subalgebras of A that is mapped by φ into itself.

Proposition 7.7. *Suppose that φ is an endomorphism of a C^*-algebra A (that is not necessarily injective), that $\mathcal{B} \subseteq A$ is a dense *-subalgebra which is φ-invariant, and that $\mathcal{B}$ is a C^*-*local *subalgebra of A.*

Let $\sigma\colon \mathbb{Z} \to \mathrm{Aut}(A_e)$ be the corresponding action of the integers $\mathbb{Z}$ on the inductive limit A_e of the sequence $A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$.

The following properties (i) and (ii) are equivalent:

(i) *For every $b_1, b_2, c \in \mathcal{B}$, and $\varepsilon > 0$ there exist $k,\, n_1,\, n_2 \in \mathbb{N} \cup \{0\}$ and elements $e_1,\, e_2 \in A$ such that, for $j \in \{1, 2\}$,*

$$\| \, e_j^* \varphi^k(b_j^* b_j)\, e_j - \varphi^{n_j}(b_j^* b_j) \,\| < \varepsilon \quad \text{and} \quad \| \, e_1^* \varphi^k(c) e_2 \, \| < \varepsilon \, . \tag{13}$$

(ii) *The action $\sigma\colon \mathbb{Z} \to \mathrm{Aut}(A_e)$ of $G := \mathbb{Z}$ on A_e is G-separating (Def. 7.4).*

Proof. Let $\mathcal{C} := \bigcup_m \varphi_m(\mathcal{B})$. Since $\mathcal{B}$ is a dense *-subalgebra of A, A_e is the closure of of $\mathcal{C}$ in $\ell_\infty(A)/c_0(A)$:

$$\mathcal{C} \subseteq \bigcup_m \varphi_m(A) \subseteq A_e \subseteq \ell_\infty(A)/c_0(A) \, .$$

Since $\varphi_m(\mathcal{B})$ is a C^*-local algebra for each $m \in \mathbb{N}$ and $\varphi_m(\mathcal{B}) \subseteq \varphi_{m+1}(\mathcal{B})$, the *-subalgebra $\mathcal{C}$ of A_e is a dense C^*-local subalgebra of A_e that satisfies $\sigma(\mathcal{C}) \subseteq \mathcal{C}$.

(i)⇒(ii): Since $\mathcal{B}$ is a dense *-subalgebra of A, we may suppose that the $e_1, e_2 \in A$ that satisfy the inequalities (13) are actually in $\mathcal{B}$ itself.

By Lemma 7.6 it suffices to show that, for $x_1, x_2, y \in \mathcal{C} \subseteq A_e$ and $\varepsilon > 0$ there exists $d_1, d_2 \in A_e$ and $k_1, k_2 \in \mathbb{Z}$ such that for $j = 1, 2$,

$$\|d_j^* x_j^* x_j d_j - \sigma^{k_j}(x_j^* x_j)\| < \varepsilon \quad \text{and} \quad \|d_1^* y d_2\| < \varepsilon \, . \tag{14}$$

Since $\mathcal{C}$ is the union of the family of images $\varphi_m(\mathcal{B}) \subseteq A_e$ of $\mathcal{B}$, there exists $m \in \mathbb{N}$ and $b_1, b_2, c \in \mathcal{B}$ with $\varphi_m(b_j) = x_j$ and $\varphi_m(c) = y$.

We apply the condition in part (i) to $(b_1, b_2, c, \varepsilon)$ and get $e_1, e_2 \in \mathcal{B}$ and $k,\, n_1,\, n_2 \in \mathbb{N} \cup \{0\}$ such that the inequalities (13) are fulfilled.

Since φ_m is a contractive linear map and $\varphi_m \circ \varphi^\ell(a) = \sigma^\ell(\varphi_m(a))$ for $a \in A$ and $\ell \in \mathbb{N}$, we get that $d_j := \sigma^{-k}(\varphi_m(e_j))$ and $k_j := n_j - k$ fulfill the inequalities (14).

(ii)⇒(i): Let $b_1, b_2, c \in \mathcal{B}$ and $\varepsilon > 0$. Since the action of $\mathbb{Z}$ defined by σ is G-separating on A_e, there exists $d_1, d_2 \in A_e$ and $k_1, k_2 \in \mathbb{Z}$ such that, for $j = 1, 2$,

$$\|d_j^* \varphi_1(b_j^* b_j) d_j - \sigma^{k_j}(\varphi_1(b_j^* b_j))\| < \varepsilon \quad \text{and} \quad \|d_1^* \varphi_1(c) d_2\| < \varepsilon \, .$$

Since $\mathcal{C}$ is a dense *-subalgebra of A_e we may suppose that $d_1, d_2 \in \varphi_\ell(\mathcal{B})$ for some $\ell \in \mathbb{N}$. Then there are $y_1, y_2 \in A$ such that $d_j = \varphi_\ell(y_j)$. Since $\varphi_1 = \varphi_\ell \circ \varphi^{\ell-1}$ and

$\sigma^n \circ \varphi_\ell = \varphi_\ell \circ \varphi^n$ for $n \geq 0$ one gets, for $x \in A$, $m := \min(0, k_1, k_2)$ and $f_j := \varphi^{-m}(y_j)$ that

$$\varphi_\ell\big(f_i^* \varphi^{\ell-1-m}(x) f_j\big) = \sigma^{-m}\big(d_i^* \varphi_1(x) d_j\big),$$
$$\sigma^{-m}\big(\sigma^{k_j} \circ \varphi_\ell \circ \varphi^{\ell-1}(x)\big) = \varphi_\ell(\varphi^{k_j-m+\ell-1}(x)).$$

This gives

$$\|\varphi_\ell\big(\, f_j^* \varphi^{\ell-1-m}(b_j^* b_j) f_j - \varphi^{k_j-m+\ell-1}(b_j^* b_j)\,\big)\| < \varepsilon, \quad \|\varphi_\ell\big(f_1^* \varphi^{\ell-1-m}(c) f_2\big)\| < \varepsilon\,.$$

Since $\|\varphi_\ell(a)\| = \lim_{n\to\infty} \|\varphi^n(a)\|$ we find sufficiently large $n \in \mathbb{N}$ such that with

$$e_j := \varphi^n(f_j), \quad k := n + \ell - 1 - m \quad \text{and} \quad n_j := n + k_j - m + \ell - 1$$

the inequalities (13) are fulfilled. □

We remind the reader of the notion of properly outer actions needed for the next theorem.

Definition 7.8 ([28])**.** Suppose that (A, G, σ) is a C^*-dynamical system and that G is discrete. The action σ will be called *element-wise properly outer* if, for each $g \in G \setminus \{e\}$, the automorphism σ_g of A is properly outer in the sense of [16, def. 2.1], i.e., $\|\, \sigma_g | I - \mathrm{Ad}(U)\, \| = 2$ for any σ_g-invariant non-zero ideal I of A and any unitary U in the multiplier algebra $\mathcal{M}(I)$ of I. See also [31, thm. 6.6(ii)].

An action σ is *residually properly outer* if, for every G-invariant ideal $J \neq A$ of A, the induced action $[\sigma]_J$ of G on A/J is *element-wise* properly outer.

Theorem 7.9. *Let $\mathcal{B} \subseteq A$, A_e, φ and σ be as in Proposition 7.7, with endomorphism $\varphi\colon A \to A$ that satisfies the above discussed conditions (ND) and (CP). Suppose that:*

(i) *For every $b_1, b_2, c \in \mathcal{B}$ and $\varepsilon > 0$ there exist k, n_1, n_2 $\in \mathbb{N} \cup \{0\}$ and elements e_1, e_2 $\in A$ such that, for $j \in \{1, 2\}$,*

$$\|\, e_j^* \varphi^k(b_j^* b_j)\, e_j - \varphi^{n_j}(b_j^* b_j)\,\| < \varepsilon\,, \quad \text{and} \quad \|\, e_1^* \varphi^k(c) e_2\,\| < \varepsilon\,. \tag{15}$$

(ii) *For every $n \in \mathbb{N}$ and every σ-invariant closed ideal $J \neq A_e$ of A_e the automorphism $([\sigma]_J)^n$ of A_e/J is properly outer.*

Then $A_e \rtimes_\sigma \mathbb{Z}$ and its hereditary C^*-subalgebra $A \rtimes_\varphi \mathbb{N}$ are strongly purely infinite.

Proof. It suffices to show that $A_e \rtimes_\sigma \mathbb{Z}$ is strongly purely infinite, because $A \rtimes_\varphi \mathbb{N}$ is naturally isomorphic to the (full) hereditary C^*-subalgebra of $A_e \rtimes_\sigma \mathbb{Z}$ that is generated by its C^*-subalgebra $\varphi_1(A)$. Hereditary C^*-subalgebras of s.p.i. algebras are again

s.p.i. by [26, Prop. 5.11]. By Proposition 7.7, the condition (i) is equivalent to the G-separation of the action σ of $G := \mathbb{Z}$ on A_e generated by the restriction of the backward shift on $\ell_\infty(A)/c_0(A)$ to A_e. The condition (ii) says that the action σ of $\mathbb{Z}$ on A_e is residually properly outer (cf. Definition 7.8). Since every abelian group is exact, the action σ is exact in the sense of [39, def. 1.2]. So all the assumptions of [28, Theorem 1.1] are satisfied for A_e and $\sigma\colon \mathbb{Z} \to \mathrm{Aut}(A_e)$. Thus, $A_e \rtimes_\sigma \mathbb{Z}$ is strongly purely infinite. □

Proof of Theorem 1.4: Follows form Theorem 7.9 as $\mathcal{B} := A$ is a φ-invariant C^*-local *-subalgebra of A. □

8. Cuntz-Pimsner algebras

An application of Theorem 7.9 to certain special Cuntz-Pimsner $\mathcal{O}(\mathcal{H})$ algebras is given by the construction below. It is implicitly contained in [19].

Let C be a stable σ-unital C^*-algebra C, and let $h\colon C \to \mathcal{M}(C)$ be a non-degenerate *-homomorphism (i.e. $h(C)C = C$) that is faithful and satisfies $h(C) \cap C = \{0\}$. Notice that h extends to a faithful strictly continuous unital *-endomorphism $\mathcal{M}(h)$ of $\mathcal{M}(C)$. To simplify notation we denote the endomorphism $\mathcal{M}(h)$ of $\mathcal{M}(C)$ again by h, unless we wish to make an emphasis on the difference between $\mathcal{M}(h)$ and h.

In the following let $\mathcal{H}(h, C)$, or simply $\mathcal{H}$, denote Hilbert bi-module given by $\mathcal{H} := C$, with right multiplication $b \mapsto ba$, left-multiplication $b \mapsto h(a)b$ and Hermitian form $\langle c, b\rangle := c^*b$. The C^*-algebra $\mathcal{L}(\mathcal{H})$ of adjoint-able bounded operators on $\mathcal{H}$ is here the same as $\mathcal{M}(C)$.

The closer look in [19] to the work of Pimsner [34] shows that, under our special assumptions on $h\colon C \to \mathcal{M}(C)$, the natural epimorphism from the Toeplitz-Pimsner algebra $\mathcal{T}(\mathcal{H})$ onto the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{H})$ is an isomorphism, *and* that $\mathcal{T}(\mathcal{H})$ is isomorphic to a crossed product $A \rtimes_\varphi \mathbb{N}$ in the following manner:

Consider the algebraic sum $\mathcal{B} := C + h(C) + h^2(C) + \cdots$. The algebra $\mathcal{B}$ is a C^*-local *-subalgebra of $\mathcal{M}(C)$, because it is the algebraic inductive limit of the C^*-algebras $C + h(C) + \ldots + h^n(C)$. Clearly $h(\mathcal{B}) \subseteq \mathcal{B}$. Let $A \subseteq \mathcal{M}(C)$ denote the norm-closure of $\mathcal{B}$ in $\mathcal{M}(C)$, and let $\varphi := h|A$. The above stated assumptions on C and $h\colon C \to \mathcal{M}(C)$ imply that the Toeplitz-Pimsner algebra $\mathcal{T}(\mathcal{H})$ is naturally isomorphic to the semi-group crossed product $A \rtimes_\varphi \mathbb{N}$.

To prove that $\mathcal{T}(\mathcal{H})$ is strongly purely infinite it suffices to show that $\mathcal{B} \subseteq A$, φ and $(A_e, \mathbb{Z}, k \mapsto \sigma^k)$ satisfy the conditions (i) and (ii) of Theorem 7.9.

It is not possible to prove the conditions (i) and (ii) with the above weak assumptions on $h\colon C \to \mathcal{M}(C)$ that we have introduced so far, because of an example $h\colon C \to \mathcal{M}(C)$ with $C = \mathrm{C}_0(X, \mathcal{K})$ (where $X := S^2 \times S^2 \times \cdots$) given by M. Rørdam [36]. His example has the property that $\mathcal{O}(\mathcal{H})$ is a stable, separable, simple nuclear C^*-algebra that contains finite and infinite projections. In particular this algebra $\mathcal{O}(\mathcal{H})$ ($\cong \mathcal{T}(\mathcal{H})$) is not purely infinite. Therefore we require now following stronger properties (i)–(iv) for $h\colon C \to \mathcal{M}(C)$:

(i) h is a non-degenerate *-monomorphism.
(ii) h is approximately unitarily equivalent in $\mathcal{M}(C)$ to its infinite repeat $\delta_\infty \circ h$.
(iii) Each $J \in \mathcal{I}(C)$ is contained in the closed ideal of C generated by $h(J)C$.
(iv) h is approximately unitarily equivalent in $\mathcal{M}(C)$ to $\mathcal{M}(h) \circ h$.

Here $\mathcal{M}(h)\colon \mathcal{M}(C) \to \mathcal{M}(C)$ denotes the the strictly continuous extension of h to a unital *-monomorphism of $\mathcal{M}(C)$ using property (i). Recall that the *infinite repeat* endomorphism $\delta_\infty\colon \mathcal{M}(C) \to \mathcal{M}(C)$ in condition (ii) is unique up to unitary equivalence in $\mathcal{M}(C)$ and is given by the strictly convergent series $\delta_\infty(b) := \sum_{n=1}^\infty s_n b s_n^*$ for $b \in \mathcal{M}(C)$, where $s_1, s_2, \ldots \in \mathcal{M}(C)$ is a sequence of isometries with $\sum_n s_n s_n^*$ strictly convergent to 1.

Definition 8.1. We say that an action $\sigma\colon G \to \mathrm{Aut}(A)$ has the *residual weak Rokhlin property*, if the center $\mathcal{Z}(A^{**})$ of A^{**} contains a projection $P \in \mathcal{Z}(A^{**})$ that satisfies:

(i) $\sigma_g(P)P = 0$ for all $g \in G \setminus \{e\}$.
(ii) The equation $r(1-q)P = 0$ implies $r(1-q) = 0$, if $q, r \in \mathcal{Z}(A^{**})$ are any $\sigma(G)$-invariant (A-)open projections.

Here we extend σ_g to a normal automorphism $\sigma_g\colon A^{**} \to A^{**}$ of A^{**}, and with $q = 0$ above we obtain the corresponding non-residual version of the definition which we call the *weak Rokhlin property* of the action σ.

Remark 8.2. A comparison of other (generalized) Rokhlin properties can be found in [39, cor. 2.22]. Using arguments from the proof of [39, thm. 2.12] it follows that topological freeness of [4, def. 1] implies topological freeness of [39, def. 1.16] that in turn gives the weak Rokhlin property (Definition 8.1) which again implies the Rokhlin* property [39, def. 2.1] from which we get element-wise proper outerness (Definition 7.8),

and all properties coincide on *commutative* C^*-algebras $A \cong \mathrm{C}_0(X)$ and G countable. It can be easily seen that the proofs of these results pass to the corresponding versions of (generalized) residual Rokhlin properties.

We outline how above properties (i)–(iv) of $h\colon C \to \mathcal{M}(C)$ imply the conditions (i) and (ii) of Theorem 7.9: We let $D := \varphi_1(C) \subseteq A_e$, then

$$\varphi_1(\mathcal{B}) = D + \sigma(D) + \sigma^2(D) + \cdots$$

and $\varphi_1 \circ \varphi = \sigma \circ \varphi_1$ for our above defined automorphism σ of A_e associated to $\varphi = h|A$. It is easy to show (cf. [19]) that A_e is the closure of the algebraic sum $\sum_{k\in\mathbb{Z}} \sigma^k(D)$ and that the closures J_n of $\sum_{k\leq n} \sigma^k(D)$ are ideals of A_e with the property that $J_n = J_n\sigma^k(D)$ for $k \geq n$, and $\sigma^n(D)A_e\sigma^n(D) = J_n$. One can use this as a dictionary to translate our conditions on $h\colon C \to \mathcal{M}(C)$ into conditions on $D \subseteq A_e$ and σ. Let $P_0 \in (A_e)^{**}$ denote the support projection of the hereditary C^*-algebra $DA_eD \subseteq A_e$. Since $DA_eD = J_0$, the projection P_0 is an open central projection of $(A_e)^{**}$. It is shown in [19] ([5]) that the conditions (i)–(iii) imply that $P := P_0 - (\sigma^{-1})^{**}(P_0)$ has the properties (i) and (ii) of Definition 8.1 for $(A_e, \mathbb{Z}, k \mapsto \sigma^k)$. Thus conditions (i)–(iii) on h imply that the $\mathbb{Z}$-action $k \mapsto \sigma^k$ has the residual weak Rokhlin property of Definition 8.1. Using Remark 8.2 we get property (ii) of Theorem 7.9.

It is a fairly elaborate work to show that $(\mathcal{B}, \varphi = h|A)$ satisfy the inequalities (15) of Theorem 7.9 if h moreover satisfies condition (iv), but deep reasonings are not needed. In this way one can see that (i)-(iv) imply the conditions (i) and (ii) of Theorem 7.9.

Remarks 8.3. Let C be a stable σ-unital C^*-algebra, and $h\colon C \hookrightarrow \mathcal{M}(C)$ a *-monomorphism.

If h satisfy properties (i)–(iv) and if C is in addition nuclear and separable, then $\mathcal{O}(\mathcal{H})$ – build from h – is a stable separable nuclear C^*-algebra that absorbs $\mathcal{O}_\infty$ tensorial, i.e.,

$$\mathcal{O}(\mathcal{H}) \cong \mathcal{O}_\infty \otimes \mathcal{O}(\mathcal{H})\,. \tag{16}$$

We do not know if the isomorphism (16) holds in case that $h\colon C \to \mathcal{M}(C)$ satisfies (i)–(iii), but C is *not* nuclear. We did not find a counter-example for the isomorphism (16) with h satisfying only (i) and (ii). The property (iii) is used in [19] for the proof of the residual proper outerness of the corresponding $\mathbb{Z}$-action on A_e. The conditions

[5] But terminology in [19] is different, e.g. the σ used in [19] is the inverse of our σ, and the h there satisfies weaker assumptions.

(iii), (iv) and the nuclearity of C play an important role in *our* verification of the isomorphism (16).

Remark 8.4. Since many strongly purely infinite nuclear C^*-algebras are Cuntz-Pimsner algebras of the type constructed in [19] and some of them are stably projectionless, our considerations are also farer going than for example the study of local boundary actions in [29], because reduced crossed products by local boundary actions is very rich of projections by [29, lem. 8], but there are important amenable strongly p.i. C^*-algebras that do not contain any non-zero projection.

Example 8.5. Let $A := \mathrm{C}_0(0,1]$, take $g \in A_+$ defined by $g(t) := \min(4(t-1/2)_+, t)$ for $t \in [0,1]$. The map $\varphi\colon\ f \in \mathrm{C}_0(0,1] \mapsto f \circ g \in \mathrm{C}_0(0,1]$ is a *-monomorphism of A that satisfies condition (ND) and does not extend to any *-endomorphism ψ of $\mathcal{M}(\mathrm{C}_0(0,1]) \cong \mathrm{C}_b(0,1]$. In particular, it does not satisfy condition (CD).

Proof. Indeed, φ satisfies property (ND), because $g(t) = t$ for $t \in [2/3, 1]$. *Suppose* that there exists is a *-homomorphism $\psi\colon \mathrm{C}_b(0,1] \to \mathrm{C}_b(0,1]$ with $\psi(f) = \varphi(f) = f \circ g$ for $f \in \mathrm{C}_0(0,1]$. Then we would get that

$$f_1(g(t))g(t) = \psi(f_1 f_0)(t) = \psi(f_1)(t)\varphi(f_0)(t) = \psi(f_1)(t)g(t)$$

for $t \in (0,1]$, $f_1 \in \mathrm{C}_b(0,1]$, and $f_0(t) := t$. Thus, $\psi(f_1)(t) = f_1(g(t))$ for all $t > 1/2$. Since $\psi(f_1)$ is a bounded continuous function on $(0,1]$, the limit $\lim_{\delta\to 0+} \psi(f_1)(1/2+\delta)$ exists and is equal to $\psi(f_1)(1/2)$. It follows that $\lim_{\varepsilon\to 0+} f_1(\varepsilon) = \lim_{\delta\to 0+} f_1(g(1/2+\delta))$ exists for all $f_1 \in \mathrm{C}_b(0,1]$. The function $f_1(t) := \sin(1/t)^2$ is in $\mathrm{C}_b(0,1]_+$ and has no limit at zero. It contradicts the existence of ψ. □

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